# Lesson III

# PONTRJAGIN MAXIMUM PRINCIPLE

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The Pontrjagin Maximum Principle (PMP, in short), elaborated during the 50's, represents a general approach to the optimal control problem, without need of restrictive assumptions such as (R1) (R2) (R3) (R4).....

We start by recalling the data

Control system: (CS) $\dot{x} = f(x, u)$

Admissible controls: piecewise continuous functions $u(t) : [0, +\infty) \to U$, where $U \subset \mathbf{R}^m$

Endpoints conditions: $x(0) = \bar{x}$ , $x(T) = \underline{x}$

Functional to be minimized: $J(T, u(\cdot)) = \int_0^T f_0(x(t), u(t))\, dt$

Let us introduce an auxiliary (scalar) variable

$$z(t) = \int_0^t f_0(x(s), u(s))\, ds$$

$z(t)$ is continuous everywhere and of class $C^1$ except possibly at the control jumps. Moreover,

$$\dot{z}(t) = f_0(x(t), u(t))$$

and $z(0) = 0$. The new variable will be formally incorporated into the state vector, which becomes $(z, x) = (z, x_1, \ldots, x_n)$

The statement of the PMP exploits the Hamiltonian formalism and hence, the introduction of adjoint variables. Let us denote by $\psi \in \mathbf{R}^n$ the adjoint variable of the (original) state variable $x$, and by $\omega \in \mathbf{R}$ the adjoint variable of the new auxiliary variable $z$. Let

$$H(x, u, \psi, \omega) = \omega f_0(x, u) + \psi \cdot f(x, u)$$

Comparing with the approach discussed in Lesson II, we see that the adjoint variable $\psi$ plays the same role as the Lagrange multiplier, and it is therefore thought of as a row-vector. This is especially convenient for a correct use of the Hamiltonian formalism.

It is straightforward to check that

$$\frac{\partial H}{\partial \psi} = f(x, u)$$

$$\frac{\partial H}{\partial \omega} = f_0(x, u)$$

Moreover, from

$$-\frac{\partial H}{\partial z} = 0 \ (= \dot{\omega})$$

we realize that $\omega$ is constant. Finally,

$$-\frac{\partial H}{\partial x} = -\psi \frac{\partial}{\partial x} f(x, u) - \omega \frac{\partial}{\partial x} f_0(x, u) \tag{1}$$

We are now ready to state the PMP.

**Theorem.** (Necessary condition for optimality).
Assume that $U$ is compact. If $(T^*, u^*(\cdot), x^*(\cdot))$ is an optimal triplet for the given problem, then there exist a real constant $\omega^* \leq 0$ and a (non vanishing) solution $\psi^*(t)$ of the "adjoint" equation

$$\dot{\psi} = -\frac{\partial H}{\partial x}(x^*(\cdot), u^*(\cdot), \psi, \omega^*) \tag{2}$$

such that

$$H(x^*(t), u^*(t), \psi^*(t), \omega^*) = \max_{u \in U} H(x^*(t), u, \psi^*(t), \omega^*) = 0 \tag{3}$$

at each instant $t \in [0, T^*]$ where the optimal control $u^*(\cdot)$ is continuous.

**Remarks**

$\diamondsuit$ By (1) and (2), the adjoint equation takes the form

$$\dot{\psi} = -\psi \frac{\partial}{\partial x} f(x,u) - \omega \frac{\partial}{\partial x} f_0(x,u) \tag{4}$$

If $x(t)$ and $u(t)$ are known, then (4) becomes an affine system of (time-varying) differential equations, for which a formula for the general integral is available (depending on a vector of constants $k \in \mathbf{R}^n$).

$\diamondsuit$ Since we are assuming that $U$ is compact, the maximum of the function

$$u \mapsto H(x, u, \psi, \omega) \tag{5}$$

exists for each $x, \psi, \omega$.

$\diamondsuit$ If the maximum in (5) is attained at an interior point of $U$, then (3) implies the stationarity condition

$$\frac{\partial H}{\partial u}(x^*(t), u^*(t), \psi^*(t), \omega^*) = 0$$

for each $t \in [0, T^*]$.

However, it is worthwhile to notice that in many practical applications, $u^*(t) \in \partial U$.

$\diamondsuit$ Formula (3) provides three types of information:

**(I1)** the function $H$ takes the maximum when it is evaluated along the optimal values of the variables;

**(I2)** $H$ is constant when it is evaluated along the optimal values of the variables; in this sense, it plays the same role as a transversality condition in CV

**(I3)** the value of $H$, when it is evaluated along the optimal values of the variables, is zero.

$\diamondsuit$ The function $H$ is homogeneous with respect to the vector of the adjoint variables $(\psi, \omega)$. If $\omega \neq 0$, it can be normalized in such a way that $\omega = -1$. This choice is usually convenient for computations but, sometimes, it can be better to normalize one of the components of $\psi$.

Terminology: sometimes, when $\omega \neq 0$ the problem is said to be "normal", in the opposite case it is said to be "abnormal"

♢ The PMP reduces an infinite dimension optimization problem to find the maximum of a function with a finite number of variables, parameterized by $t$ and with the possible exception of a finite number of points.

♢ In general, the PMP is only a necessary condition for optimality. However, it can be used to select "candidate" optimal controls.

A possible procedure is:

◦ solve the (cascade) system of ODE formed by (CS) and (4); we expect that the general integral depends on a vector of constants $(c, k) \in \mathbf{R}^{2n}$, as well as on the parameters $\omega$ and $u$

◦ Apply the controllability conditions, in order to eliminate $c$ and $k$

◦ Apply (3) in the sense of **(I1)** to determine for each $t$ the value to be assigned to $u^*(t)$ (except for a finite number of points)

◦ Use again (3) in the sense of **(I2)(I3)** to determine $T^*$.

♢ In general it may be very hard to follow the aforementioned procedure. However, there are fortunate exceptions: for instance, when the control system is linear

$$\dot{x} = Ax + Bu$$

and the integrand of the cost functional does not depend on $x$, that is $f_0(x, u) = f_0(u)$. Under these hypothesis, the system (CS) + (4) is linear.

♢ For the minimum time problem, the Hamiltonian function becomes

$$H(x, u, \psi, \omega) = \omega + \psi \cdot f(x, u)$$

and the PMP states that

$$H(x^*(t), u^*(t), \psi^*(t), \omega^*) = \omega^* + \psi^*(t) \cdot f(x^*(t), u^*(t)) = 0$$

Moreover, for the time optimal problem we always have $\omega^* \neq 0$, otherwise, $\psi^*(t) = 0$ which is not allowed for the PMP.

$\diamondsuit$ The PMP remains a valid necessary condition for problems with a fixed final time $T$, in the sense of **(I1)** and **(I2)** . However, in this case it is not possible to predict the value of the constant $H(x^*(t), u^*(t), \psi^*(t), \omega^*)$. On the other hand, we have one less unknown to determine.

$\diamondsuit$ Sometimes, the PMP can be used as a sufficient condition as well, when combined with other facts: for instance when existence and uniqueness of solutions has been independently proved.