Lesson IV

TWO EXAMPLES

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Mathematical and physical methods for space sciences

Zermelo navigation problem.
Chose coordinates in the plane in such a way that the cost coincides with the x axis and the sea with the half-plane $y > 0$. Assume that the current is parallel to the cost (in the positive direction) and its intensity proportional to the distance from the cost, say cy ($c > 0$). Set the modulus of the velocity of the ship $v = 1$.

Assume that the ship departs from the origin, and the arrival harbor is in $(a, 0)$, with $a > 0$. Find the optimal time trajectory ($f_0 = 1$).

$$\text{Mathematical model :} \quad \begin{cases} \dot{x} = cy + \cos u \\ \dot{y} = \sin u \end{cases} \tag{1}$$

where $u \in [0, 2\pi]$ denotes the angle formed by the velocity vector and the x axis.

The Hamiltonian function is

$$H(x, y, u, \psi_1, \psi_2, \omega) = \omega + \psi_1(cy + \cos u) + \psi_2 \sin u$$

Of course, computing the partial derivatives of H w.r.t. ψ_1 and ψ_2 we recover the dynamics equations. Moreover,

$$\begin{cases} \dfrac{\partial H}{\partial x} = 0 (= \dot{\psi}_1) \\ \\ \dfrac{\partial H}{\partial y} = c\psi_1 (= -\dot{\psi}_2) \end{cases} \qquad \text{(adjoint equations)}$$

and finally

$$\frac{\partial H}{\partial u} = -\psi_1 \sin u + \psi_2 \cos u = 0 \qquad \text{(stationarity condition).}$$

The general integral of the adjoint system is

$$\begin{cases} \psi_1(t) = k_1 \\ \psi_2(t) = k_2 - ck_1 t \end{cases} \tag{2}$$

where k_1, k_2 are arbitrary constants.

The stationarity condition becomes

$$-k_1 \sin u + (k_2 - ck_1 t) \cos u = 0 \tag{3}$$

We can see that $k_1 \neq 0$, otherwise $u = \pi/2$ and it would impossible to met the controllability condition.

Assuming no constraints on the orientation of the rudder, we can solve (3) w.r.t. u in order to find candidate optimal controls. We have

$$u = \operatorname{arctg}\left(\frac{k_2}{k_1} - ct\right)$$

Replacing in (1), we obtain a linear system with a time-varying forcing term

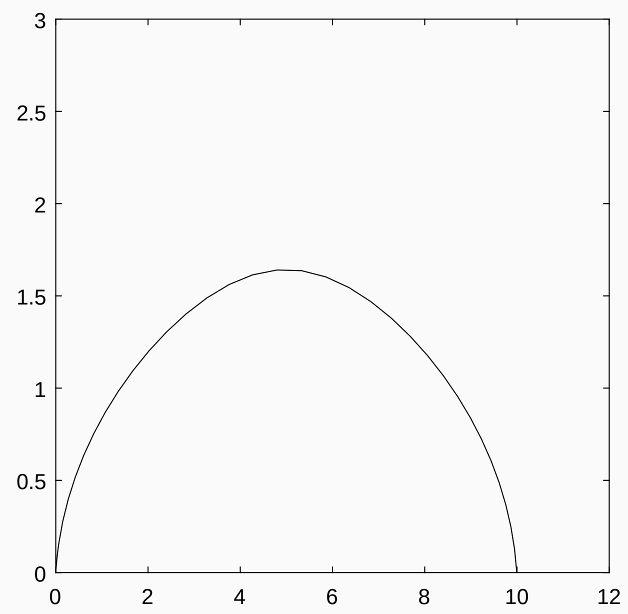
$$\begin{cases} \dot{x} = cy + \cos\left[\mathrm{arctg}\left(\frac{k_2}{k_1} - ct\right)\right] \\ \\ \dot{y} = \sin\left[\mathrm{arctg}\left(\frac{k_2}{k_1} - ct\right)\right] \end{cases}$$

The general integral of this system depends on four constants (k_1, k_2 plus two integration constants). In addition, we still have the unknowns ω and T^*. Because of the homogeneity of H, one constant can be fixed: in this problem, the more convenient choice is perhaps $k_1 = 1$. The remaining constants can be determined by the aid of the endpoints conditions

$$x(0) = y(0) = 0\,, \quad x(T^*) = a\,, \quad y(T^*) = 0$$

and recalling that $H = 0$ when computed along an optimal solution.

In spite of the complicate form of the forcing term, the general integral can be explicitly obtained by appropriate tricks (re-parametrization etc.).



The shape of the optimal trajectory (see the Figure) has been obtained by MATLAB simulation, with $c = 2$, $a = 10$. The approximate value of T^* is 4.167.

The simulation also shows that ω^* remains (reasonably) constant along the trajectory.

Vertical landing. Consider a body (of mass $= 1$) in a vertical gravitational field with constant g ($0 < g < 1$). Let us denote by x the hight of the body at a generic instant and let 0 be the ground level. Let finally $x(0) = s > 0$ and $\dot{x}(0) = v < 0$ be the initial values of the height and speed (resp.). If the fall is free, according to the Newton gravitation law, the dynamic equation is

$$\ddot{x} = -g$$

which leads to

$$\dot{x}(t) = -gt + v \ , \quad x(t) = -\frac{gt^2}{2} + vt + s$$

We see that the body will "land" at time

$$T = \frac{v + \sqrt{v^2 + 2gs}}{g}$$

At the impact instant, the modulus of the velocity will be strictly positive.

Assume now that we can slow down the fall by the action of a rocket. The equation should be modified accordingly

$$\ddot{x} = -g + u \tag{4}$$

where u represents the thrust and it is interpreted as a control. It is natural to assume that u is constrained, say $0 \leq u \leq 1$.

We want to find u in order to achieve safe landing, that is

$$x(T) = \dot{x}(T) = 0$$

(controllability conditions) in minimum time.

System (4) is a very approximative model. For instance, in the case of Moon landing, the fuel consumption implies a reduction of the mass which cannot be neglected. A most realistic model requires a third equation, which describes the mass change.

We test two different strategies.

Strategy 1. u= constant. To met the controllability conditions we must have

$$u = g + \frac{v^2}{2s} \quad \text{and} \quad T_1 = -\frac{2s}{v}$$

Note that safe landing is possible only if $s \geq \frac{v^2}{2(1-g)}$ (reachable set).

Strategy 2. Bang-Bang control (with only one switch). Let

$$u(t) = \begin{cases} 0 & 0 \le t \le \tau \\ 1 & \tau \le t \le T_2 \end{cases} \tag{5}$$

where $\tau > 0$ has to be determined. The integration of the system (4) must be now performed in two steps, updating the "initial" conditions at the second step. Imposing the controllability conditions, we find

$$\tau = \frac{v + \sqrt{(1-g)(v^2 + 2gs)}}{g} \quad \text{and}$$

$$T_2 = \frac{1}{g}\left[v + \frac{\sqrt{(1-g)(v^2 + 2gs)}}{1-g}\right]$$

A direct comparison shows that $T_2 < T_1$

We now check that the second strategy mets the PMP. Rewritten as a first order system, the dynamics equations are

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g + u \end{cases} \tag{6}$$

$(x = x_1)$ that is $f(x_1, x_2, u) = \begin{pmatrix} x_2 \\ u - g \end{pmatrix}$

Since this is a minimum time problem, $f_0 = 1$. Hence, the Hamiltonian function is

$$H(x_1, x_2, u, \psi_1, \psi_2, \omega) = \omega + \psi_1 x_2 + \psi_2 (u - g)$$

The system of the adjoint equations is

$$\begin{cases} \dot{\psi}_1 = 0 \\ \dot{\psi}_2 = -\psi_1 \end{cases} \tag{7}$$

whose general integral is

$$\begin{cases} \psi_1(t) = k_1 \\ \psi_2(t) = k_2 - k_1 t \end{cases} \tag{8}$$

Chose for k_1^*, k_2^*, ω^* negative values in such a way that

$$k_1^* v - k_2^* g = -\omega^* \quad \text{and} \quad \frac{k_2^*}{k_1^*} = \tau \tag{9}$$

and let $\psi_1^*(t) = k_1^*$, $\psi_2^*(t) = k_2^* - k_1^* t$. We have

$$H(x_1, x_2, u, \psi_1^*(t), \psi_2^*(t), \omega^*) = \omega^* + k_1^* x_2 - (k_2^* - k_1^* t) g + (k_2^* - k_1^* t) u$$

Now it is clear that the maximum of H is attained when

$$u = \begin{cases} 0 & \text{if } k_2^* - k_1^* t < 0 \\ 1 & \text{if } k_2^* - k_1^* t > 0 \end{cases} \tag{10}$$

By virtue of (9), we can finally check that $H = 0$