Lesson V

THE REGULATOR PROBLEM

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MASTER UNITO, 2020-21

Mathematical and physical methods for space sciences

Consider a linear time-invariant system

$$\dot{x} = Ax + Bu \tag{1}$$

with $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^m$.

Admissible controls will be now all the piecewise continuous functions $u(t) : [0, +\infty) \to \mathbf{R}^m$ (no constraints on the values of $u(t)$).

As usual, we denote $x(t) = x(t; \bar{x}, u(\cdot))$ the solution of (1) corresponding to the initial state $\bar{x}$ and an admissible control $u(t)$.

Let us associate to (1) the cost functional

$$J(\bar{x}, u(\cdot)) = \frac{1}{2}\int_0^{+\infty} (||x(t)||^2 + ||u(t)||^2)\, dt \qquad (2)$$

Note that the integrand of the cost functional is convex, and $J(\bar{x}, u(\cdot)) \geq 0$ for each $\bar{x}$ and $u(\cdot)$

The *linear-quadratic optimal regulator problem on the infinite horizon* (in short, LQR problem) can be formulated in the following way.

For any initial state $\bar{x}$ find (if any) an admissible control $u(t)$ in such a way that the cost functional $J(\bar{x}, u(\cdot))$ takes the minimum possible value.

It is possible to state a similar problem on the finite horizon, taking the integral on an interval $[0, T]$ for a fixed $T > 0$. In this case, a final endpoint condition $x(T) = \underline{x}$ should be assigned. It is also possible to consider more general forms of the functional.

The stabilization problem.

Definition. We say that system (1) is *asymptotically controllable* if for each $\bar{x} \in \mathbf{R}^n$ there exists a control $u_{\bar{x}}(t)$ such that for the corresponding solution $x(t)$ of the problem

$$\begin{cases} \dot{x} = Ax + Bu_{\bar{x}}(t) \\ x(0) = \bar{x} \end{cases} \tag{3}$$

we have $\lim_{t \to +\infty} x(t) = 0$.

Definition. We say that (1) is *stabilizable* if there exists a matrix F with m rows and n columns such that for each $\bar{x} \in \mathbf{R}^n$ the corresponding solution $x(t)$ of the problem

$$\begin{cases} \dot{x} = (A + BF)x \\ x(0) = \bar{x} \end{cases} \tag{4}$$

satisfies $\lim_{t \to +\infty} x(t) = 0$.

(4) results from (1) by the substitution $u = Fx$.
The function $u = Fx$ is called a (static state) stabilizing feedback.

Theorem. The following conditions are equivalent.
(i) System (1) is asymptotically controllable.
(ii) System (1) is stabilizable.
(iii) There exists a symmetric, positive definite matrix P such that

$$A^{\mathrm{t}}P + PA - PBB^{\mathrm{t}}P = -I \ . \tag{5}$$

In this case, a stabilizing feedback is obtained setting $u = Fx$ with $F = -\alpha B^{\mathrm{t}}P$ and $\alpha \geq \frac{1}{2}$.

(5) is called the Algebraic Matrix Riccati Equation, in the matrix unknown P. Note that such equation is nonlinear.
In the set of symmetric, positive definite matrices, (5) admits at most one solution.

Convergence of the cost functional.

Since the cost functional is expressed by means of an improper integral, the problem is nontrivial only if

$$(\mathbf{C}) \qquad \forall \bar{x} \in \mathbf{R}^n \ \exists u_{\bar{x}}(t) \ : \quad J(\bar{x}, u_{\bar{x}}(\cdot)) < +\infty$$

Indeed, if ($\mathbf{C}$) holds then the minimum of the cost functional (if it exists) is finite.

Proposition 1. Assume that condition ($\mathbf{C}$) holds. Then, for the solution $x(t) = x(t; \bar{x}, u_{\bar{x}}(\cdot))$ of (1) we have

$$\lim_{t \to +\infty} x(t) = 0$$

i.e., system (1) is asymptotically controllable.

The converse of the previous proposition is true, as well.

Proposition 2. Assume that system (1) is asymptotically controllable. Then, for each initial state $\bar{x}$ there exists a control $u_{\bar{x}}(t) : [0, +\infty) \to \mathbf{R}^m$ such that the integral in (2) converges i.e., $(\mathbf{C})$ holds.

With respect to the LQR problem, the asymptotic controllability property can be reviewed as a generalized final endpoint condition.

Proposition 2 states that, under the asymptotic controllability hypothesis, the LQR problem is nontrivial, <u>but not yet</u> that a solution exists.

Necessary condition for optimality.

The Hamiltonian function, for our problem is

$$H(x,u,\psi,\omega) = \frac{\omega}{2}(||x(t)||^2 + ||u(t)||^2) + \psi \cdot (Ax + Bu)$$

which yields

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \psi} = Ax + Bu \\ \\ \dot{\psi} = -\dfrac{\partial H}{\partial x} = -\omega x^{\mathbf{t}} - \psi A \quad \text{(adjoint equation)} \end{cases} \tag{6}$$

(recall that ψ is a row-vector).

Now, assume that for a given initial state $\bar{x}$ an optimal control $u^*(t)$ is known, and let $x^*(t)$ be the corresponding optimal trajectory.

Since there is no constraints, we must have

$$\frac{\partial H}{\partial u} = \omega^* u^*(t)^{\mathbf{t}} + \psi^*(t) B = 0 \tag{7}$$

for each $t > 0$, some $\omega^* \leq 0$ and some nontrivial solution $\psi^*(t)$ of the adjoint equation where ω and x are replaced by ω^* and $x^*(t)$ (this is guaranteed by the PMP).

(7) shows that $\omega^* \neq 0$: otherwise, also $\psi^*(t) = 0$. Thus, from now on we can take $\omega^* = -1$. (7) also implies

$$u^*(t)^{\mathbf{t}} = \psi^*(t) B \tag{8}$$

The information provided by the PMP can be summarized in the following necessary condition for optimality.

Proposition 3. Let $u^*(t)$ be an optimal control for the initial state $\bar{x}$ and let $x^*(t)$ be the corresponding optimal trajectory. Let $\psi^*(t)$ be the solution of the adjoint equation (with $\omega = -1$ and $x = x^*(t)$), provided by the PMP. Then, the pair $(x^*(t), \psi^*(t))$ is a solution of the system

$$\begin{cases} \dot{x} = Ax + BB^{\mathbf{t}}\psi^{\mathbf{t}} \\ \dot{\psi} = x^{\mathbf{t}} - \psi A \end{cases} \tag{9}$$

Existence and construction of the solution.

On the base of Propositions 1 and 2, we can limit the discussion to systems which satisfy the asymptotic controllability condition (or one of its equivalent forms exposed above).

Proposition 4. Assume that (1) is asymptotically controllable, and let P be the symmetric and positive definite solution of the matrix equation (5). Let $x(t)$ be the solution of the system

$$\dot{x} = Ax - BB^{\mathbf{t}}Px \tag{10}$$

such that $x(0) = \bar{x}$, and let $\psi(t) = -x(t)^{\mathbf{t}}P$.
Then, the pair $(x(t), \psi(t))$ is a solution of (9), and $\lim_{t\to\infty}(x(t), \psi(t)) = (0, 0)$.

(the proof exploits (5)).
Note that in (10) the variable ψ does not appear anymore.

Proposition 4 states that solving (10) with the initial condition $\bar{x}$ and setting $\psi(t) = -x(t)^{\mathbf{t}}P$, we are able to construct, according to (8), a control law which satisfies the necessary condition (9). We therefore obtain a candidate for optimality.

It is possible to prove by direct arguments that such a control is really an optimal control for the LQR problem at the given initial state.

Theorem. Assume that (1) is asymptotically controllable, and let P be the symmetric and positive definite solution of the matrix equation (5). Let $x(t)$ be the solution of equation (10) corresponding to the initial state $\bar{x}$. Then, the LQR problem associated to (1) has a unique solutions, given by

$$u^*(t) = -B^{\mathbf{t}} P x(t)$$

This Theorem shows in particular that:

• the solution of the LQR problem can be realized in feedback form $u = -B^{\mathbf{t}} P x$

• the asymptotic controllability assumption is also sufficient for the existence of a (unique and finite) solution

$$\text{Existence} + \text{finiteness} \Rightarrow (\mathbf{C}) \Leftrightarrow \text{AC}$$

$$\Downarrow$$

$$\text{Existence} + \text{finiteness} \Leftarrow (\mathbf{C}) + \text{existence}$$