Basics in Celestial Mechanics - L1

- Central force fields
- kepler problem
- N-body problem
- Restricted 3-body problem

Short history

After the Middle-Age ...

- Copernicus N. (Polland 1473-1543) : back to eliocentric model
- Brahe T. (Denmark 1546-1601) : first astronomic observatory → 1st research institute (geocentric model)
- kepler J. (Germany 1571-1630) : study in Tübingen, then work for Brahe → observation of planetary motions → 3 laws to describe them
- Galilei G. (Italy 1564-1642) : Principle of Inertia

  "without external forces, a body stays or moove with uniform rectilinear motion".

  1st kepler law  + Inertia Pr. $\Big\} \Rightarrow$ external forces act on planets

- Newton I. (England 1642-1727) : rigorous proofs for kepler's laws

$$\underline{F} = -\frac{GMm}{|\underline{x}|^2}\frac{\underline{x}}{|\underline{x}|}$$

$$\underline{F} = m\ddot{\underline{x}}$$

M = mass of the Sun

m = " " planet

G = const.

$\underline{x} = \underline{x}(t)$ = planet's position (Sun in $\underline{O}$)

(N) $\boxed{\ddot{\underline{x}}(t) = -\dfrac{GM}{|\underline{x}(t)|^2}\dfrac{\underline{x}(t)}{|\underline{x}(t)|}}$ acceleration of the planet due to the presence of the Sun in $\underline{0}$

in: Philosophiae Naturalis Principia Mathematica (1687)

Newton states the principles of Newtonian Mechanics.

- Beginning '900 : Einstein theory of relativity
  → Newtonian mechanics is just an approximation of the realty; it is not reliable when
  - the velocities of the objects are are close to the speed of light (special theory of Rel.)
  - we are in space rigions with hight curvature (for instance, when we are close to a big mass → Es. Mercury motion) (general theory of Rel.)
  - working in athomic or subatomic scales (Quantum Mech.)

$\boxed{\text{We will not take into accunt relativistic effects and we'll deal with "big" (w.r.to athoms) and "slow" (w.r. to light) masses mooving in Euclidean spaces.}}$

CENTRAL FORCE FIELDS

$$\boxed{\begin{aligned} &F(\underline{x}) = f(|\underline{x}|)\frac{\underline{x}}{|\underline{x}|} \\ &F: \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \text{ continuous} \\ &f: (0,+\infty) \to \mathbb{R} \text{ continuous} \end{aligned}}$$

(CF) $$\boxed{\ddot{x}(t) = f(|x(t)|)\frac{\underline{x}(t)}{|\underline{x}(t)|}}$$

- $f(r) < 0 \Rightarrow$ attractive force
- $f(r) > 0 \Rightarrow$ repulsive

Prop. Let $\underline{x}: I \subseteq \mathbb{R} \to \mathbb{R}^3 \setminus \{\underline{0}\}$ be a sol. of (CF). Then:
- $\underline{x}(t+c)$ solves (CF), $\forall c \in \mathbb{R}$ TRANSL. INVAR.
- $\underline{x}(-t)$ " " TIME REVERSE
- $A\,\underline{x}(t)$ " " ISOMETRY INVAR.

$\forall A \in O(3) = \{A \in M_3(\mathbb{R}) : A^T A = A A^T = Id_3\}$

Ex. (N) is central with $f(r) = -\frac{\mu}{r^2}$

Also $f_\alpha(r) = -\frac{\mu}{r^{1+\alpha}}$ $\alpha > 0$ (generalized kepler)

In this case $\underline{x}(t) = r(\cos(\omega t), \sin(\omega t), 0)$

is a planar sol.: $\ddot{\underline{x}} = -\omega^2 \underline{x}$

$$\overset{\|}{-\mu}\frac{\underline{x}}{r^{2+\alpha}} \iff \omega^2 = \frac{\mu}{r^{2+\alpha}}$$

$$\omega = \sqrt{\mu}\, r^{-\frac{2+\alpha}{2}}$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\mu}} r^{\frac{2+\alpha}{2}}$$ ($\alpha = 1$: 3^{rd} kepler law for circular motions)

A central field is conservative

i.e. $\exists U : \mathbb{R}^3 \setminus \{\underline{0}\} \to \mathbb{R} : \; F(\underline{x}) = \nabla U(\underline{x})$ ← POTENTIAL

and the total energy is conserved

$$= \frac{1}{2}|\dot{\underline{x}}(t)|^2 - U(\underline{x}(t))$$

<u>indeed</u>:

$\forall r_0 > 0, \; U_{r_0}(\underline{x}) = \int_{r_0}^{|x|} f(\rho)\,d\rho$ is a potential

- Let U be a potential of a c.f.f. then $U = U(|\underline{x}|)$.

Ex. $f(r) = -\dfrac{\mu}{r^{1+\alpha}} \Rightarrow U_{r_0}(r) = \dfrac{\mu}{\alpha}\left(\dfrac{1}{r^\alpha} - \dfrac{1}{r_0^\alpha}\right)$

- It is natural to select a special U by imposing a "normalizing condition" for instance: $\lim_{r \to +\infty} U(r) = 0$

$$\Rightarrow U(r) = \frac{\mu}{\alpha r^\alpha}$$

$$h = \frac{1}{2}|\dot{\underline{x}}(t)|^2 - \frac{\mu}{\alpha|\underline{x}(t)|^\alpha}$$ is conserved along a solution

- Depending on the sign of h, motions will be bounded or not:

$$h \geq -\frac{\mu}{\alpha[r(t)]^\alpha}, \quad r(t) = |\underline{x}(t)|$$

- $h \geq 0 \Rightarrow$ no bounds
- $h < 0 \Rightarrow \dfrac{\mu}{\alpha r^\alpha} \geq -h, \quad r^\alpha \leq C_h$ Hill's region

Conservation of the angular momentum

$\underline{c}(t) = \underline{x}(t) \wedge \dot{\underline{x}}(t)$ (Induced by the invariance of a central force field w.r.t. the group of rotations)

indeed: $\dot{\underline{c}}(t) = \cancel{\dot{\underline{x}}(t) \wedge \dot{\underline{x}}(t)} + \cancel{\underline{x}(t) \wedge \underbrace{\ddot{\underline{x}}(t)}_{/\!/ \underline{x}(t)}} = 0$

$\Rightarrow \underline{c}(t) = \underline{c}$ const. vector $\Rightarrow$ planar motions

RK. if $\underline{c} = \underline{0} \Rightarrow \exists\, \lambda : I \subseteq \mathbb{R} \to \mathbb{R}$:

$\dot{\underline{x}}(t) = \lambda(t)\, \underline{x}(t)$

and $\dfrac{\underline{x}(t)}{|\underline{x}(t)|} = \underline{v} \in S^1$

$\Rightarrow$ the motion is 1-dim.

$\Rightarrow$ we can fix $d = 2$

- The law of conservation of the angular momentum was discovered by Kepler (observation of Mars).

Sectorial velocity and Kepler 2nd law

Polar coordinates: let $\begin{cases} \underline{x} : I \subseteq \mathbb{R} \to \mathbb{R}^2 \setminus \{\underline{0}\} \text{ contin.} \\ t \mapsto \underline{x}(t) \end{cases}$

$\exists!\; r(t), \vartheta(t) : \quad \underline{x}(t) = r(t) \left(\cos\vartheta(t), \sin\vartheta(t) \right)$

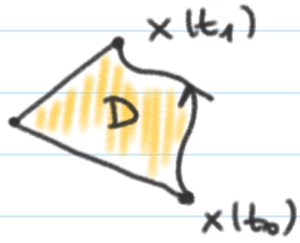
$\vartheta(t)$ up to $2k\pi$ - translations

Assume $t \in [t_0, t_1]$ so that
- $\dot{\vartheta}(t) > 0 \;\; \forall t$ (otherwise time-reverse)
- $\vartheta(t_1) - \vartheta(t_0) < 2\pi$

let: $D = \{ s\underline{x}(t) : s \in [0,1], t \in [t_0, t_1] \}$

Divergence Theorem in polar coord.:

$$\text{Area}(D) = \frac{1}{2}\int_{t_0}^{t_1} r^2(t)\dot{\theta}(t)\,dt$$



Since in polar coordinates

$$|\underline{c}| = r^2(t)\dot{\theta}(t)$$

then

$$\boxed{\text{Area}(D) = \frac{1}{2}(t_1 - t_0)|\underline{c}|}$$

2^{nd} Kepler law: in equal time the radius vector sweeps out equal areas

The law of conservation of angular momentum lets us to reduce a planar central force problem to a problem with 1 degree of freedom.

Indeed:

$$\begin{cases} \underline{x}(t) = r(t)\,\underline{e}_r \\ \dot{\underline{x}}(t) = \dot{r}(t)\,\underline{e}_r + r(t)\dot{\theta}(t)\,\underline{e}_\theta \\ \ddot{\underline{x}}(t) = \left(\ddot{r}(t) - r(t)\dot{\theta}^2(t)\right)\underline{e}_r + \left(2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\theta}(t)\right)\underline{e}_\theta \end{cases}$$

where:

$$\underline{e}_r = (\cos\theta(t), \sin\theta(t))$$

$$\underline{e}_\theta = (-\sin\theta(t), \cos\theta(t))$$

since $\ddot{\underline{x}}(t) = U'(r)\,\underline{e}_r$ we compare to obtain:

$$\begin{cases} \ddot{r}(t) - r(t)\dot{\theta}^2(t) = U'(r) \\ 2\dot{r}(t)\dot{\theta}(t) + r(t)\ddot{\theta}(t) = 0 \end{cases}$$

Since $\dot{\theta}(t) = \frac{|\underline{c}|}{r^2(t)} \Rightarrow \ddot{r} = \underbrace{\frac{|\underline{c}|^2}{r^3} + U'(r)}_{V'(r)}$

with $\boxed{V(r) = U(r) - \frac{|\underline{c}|^2}{2r^2}}$ effective potential

Conclusion: $r(t)$ satisfies $\ddot{r}(t) = V'(r(t))$ (r)

with $V(r)$ effective potential.

Fixed $|c| \geq 0$, we can solve (r), hence replace in $\dot{\theta} = \frac{|\underline{c}|}{r^2(t)}$ (θ)

and compute $\theta(t) = \theta_0 + \int_{t_0}^{t} \frac{|\underline{c}|}{r^2(s)} ds$

RK: The energy of $r(t)$ is:

$(h)_r \qquad h_r = \frac{1}{2}\dot{r}^2(t) - V(r)$

Since $\frac{1}{2}|\underline{\dot{x}}(t)|^2 = \frac{1}{2}\dot{r}^2(t) + \frac{1}{2}r^2(t)\dot{\theta}^2(t)$

$= \frac{1}{2}\dot{r}^2(t) + \frac{|\underline{c}|^2}{2r^2(t)}$

we have that $h_r = h$ (total energy).

From the energy relation: $\dot{r}(t) = \sqrt{2(h+V(r))}$

Since $\dot{\theta} = \frac{|\underline{c}|}{r^2(t)} \Rightarrow \frac{d\theta}{dr}\dot{r}(t) = \frac{|\underline{c}|}{r^2(t)}$

$\Rightarrow \frac{d\theta}{dr} = \frac{|\underline{c}|/r^2}{\sqrt{2(h+V(r))}}$

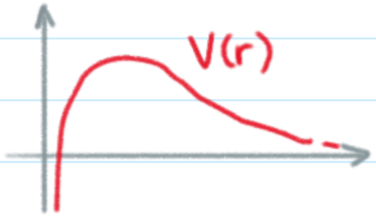
the angle ϑ increases in r

$$\Rightarrow \boxed{\vartheta(r) = \int \frac{|\underline{c}|/r^2}{\sqrt{2(h+V(r))}}\, dr}$$

equation of the orbit in polar coordinates

EX. If $U(r) = \frac{\mu}{r}$ (Kepler)

then $V(r) = \frac{\mu}{r} - \frac{|\underline{c}|^2}{2r^2}$

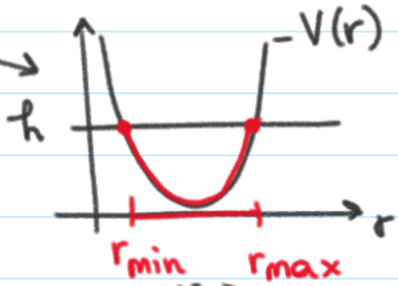


Bounded and closed orbits in a central field

The energy relation possibly imposes a bound on the radial variable:

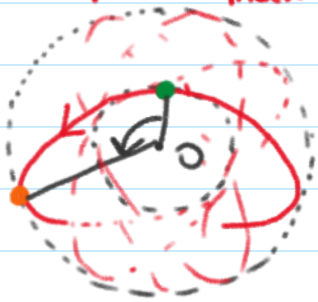
$$h = \frac{1}{2}\dot{r}^2(t) - V(r) \geq -V(r)$$

if this is the case the motion takes place in annular regions of the form



$$r_{min} \leq r \leq r_{max}$$

pericentral points — apocentral points



and $\Theta = \int_{r_{min}}^{r_{max}} \frac{|\underline{c}|/r^2}{2(E-V(r))}\, dr$ is the angle spanned (monotonic) between a pericenter and an apocenter.

ORBITS are CLOSED (periodic) iff $\Theta = \frac{m}{n} \cdot 2\pi \quad (m, n \in \mathbb{N}^*)$

(if not the orbit is everywhere dense in the annulus)

Bertrand's Theorem. There are only two cases in which **all** bounded orbits in a central field are closed:

$$U(r) = \frac{k}{r} \quad \text{and} \quad U(r) = -kr^2$$

kepler — harmonic oscillator

[For a proof see Arnold pag. 37-38]

Rectilinear motions in central force field

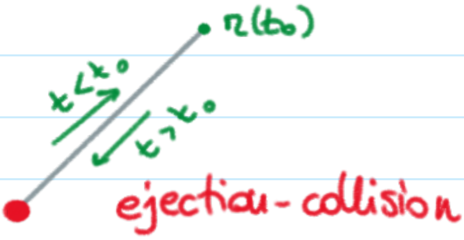
$$\underline{c} = \underline{0} \implies \underline{x}(t) = r(t)\,\underline{v}\,, \quad \underline{v} \in S^1$$

replacing $\underline{x}(t) = r(t)\,\underline{v}$ in $\ddot{\underline{x}}(t) = f(|\underline{x}(t)|)\,\frac{\underline{x}(t)}{|\underline{x}(t)|}$ we obtain

$$\boxed{\ddot{r}(t) = f(r(t))} \qquad (r)$$

Prop. Let $\underline{x}(t)$ be a 1-dim. sol. of a central force field eq. ($\underline{c} = \underline{0}$) on (α, ω) (maximal interval). If $f(r) < 0$, $\forall r$ then one of the following situations occur:

(i) α, ω both bounded and $\lim_{\substack{t \to \alpha^+ \\ t \to \omega^-}} r(t) = 0$

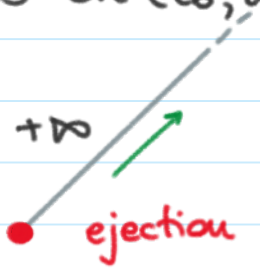


$\exists t_0 \in (\alpha, \omega)$: $\dot{r}(t) > 0$ on (α, t_0)
$\dot{r}(t_0) = 0$
$\dot{r}(t) < 0$ on (t_0, ω)

(ii) α bounded, $\omega = +\infty$

and $\lim_{t \to \alpha^+} r(t) = 0$, $\lim_{t \to +\infty} r(t) = +\infty$

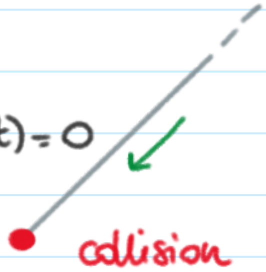
$\dot{r}(t) > 0$, $\forall t \in (\alpha, +\infty)$



(iii) $\alpha = -\infty$, ω bounded

and $\lim\limits_{t\to-\infty} r(t) = +\infty$, $\lim\limits_{t\to\omega^-} r(t) = 0$

$\dot{r}(t) < 0, \ \forall t \in (-\infty, \omega)$



PROOF. Since $\dot{r}$ is $\mathcal{C}^1(\alpha,\omega)$, then $\dot{r}(t)$

(A) has constant sign on (α,ω);

(B) admits t_0 : $\dot{r}(t_0)$

We prove that if (B) holds, then situation (i) occur; while (A) splits into cases (ii) and (iii).

(i) Assume (B), $\exists\, t_0 \in (\alpha,\omega) : \dot{r}(t_0) = 0$.

Consider $[t_0,\omega)$; we claim $\omega < +\infty$.

Since $f(r) < 0 \Rightarrow \ddot{r}(t) < 0 \Rightarrow \dot{r}(t) \searrow$

$$\Rightarrow \dot{r}(t) < 0 \quad \forall t > t_0$$

Let $\delta > 0$: $t_0 + \delta < \omega$ and $\underset{(<0)}{k} := \dot{r}(t_0+\delta)$

then :

$$r(t) = r(t_0+\delta) + \int_{t_0+\delta}^{t} \dot{r}(s)\,ds$$

$$< \underbrace{r(t_0+\delta) + k[t - (t_0+\delta)]}$$

$$\lim_{t\to\omega^-} r(t) = \lim_{t\to\omega^-} \quad \Rightarrow \omega < +\infty.$$

(otherwise $r(t) \to -\infty$)

We now proove that : $\lim\limits_{t\to\omega^-} r(t) = 0$.

The limit exists since $r(t) \searrow$. If it is $\ell > 0$ then $\lim\limits_{t\to\omega^-} f(r(t)) = f(\ell)$ and $[t_0,\omega)$ is not maximal.

We conclude case (i) arguing similarly for $\alpha > -\infty$.

Assume now (A), in particular $\dot{r}(t) > 0, \forall t$.

(ii) $\ddot{r}(t) < 0 \Rightarrow \dot{r}(t) \downarrow \Rightarrow \lim_{t \to \omega^-} \dot{r}(t)$ exists finite

If $\omega < +\infty$ then: $\underset{\substack{\downarrow \\ \text{well defined}}}{r(\omega)} = r(t_0) + \int_{t_0}^{\omega} \dot{r}(s)\,ds$

$\Rightarrow$ the solution can be extended $\Rightarrow \omega = +\infty$

Since $\dot{r}(t) > 0$, then $r(t) \uparrow$ and $\lim_{t \to +\infty} r(t)$ exists.

If $\lim_{t \to +\infty} r(t) = \ell$ then
- $\exists (t_n)_n : \dot{r}(t_n) \longrightarrow 0$
- $\lim_{t \to +\infty} \ddot{r}(t) = f(\ell) < 0$

for n suffic. large: $\dot{r}(t_n) = \dot{r}(\bar{t}) + \int_{\bar{t}}^{t_n} \ddot{r}(s)\,ds$

$$\le \dot{r}(\bar{t}) + \underbrace{(t_n - \bar{t})}_{\to +\infty \text{ if } n \to +\infty} \underbrace{\frac{f(\ell)}{2}}_{<0}$$

$\ddot{r}(s) \le \frac{f(\ell)}{2} \quad \forall s \ge \bar{t}$

$\longrightarrow -\infty$ contrad. with $\dot{r}(t_n) \to 0$

With similar arguments we prove that:
$\alpha > -\infty$ and $\lim_{t \to \alpha^+} r(t) = 0$.

The proof of (iii) is similar to (ii). ∎

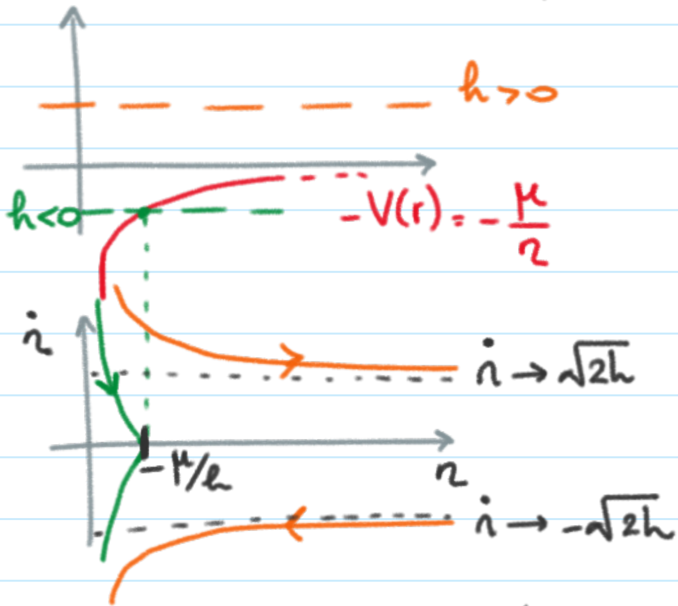
1d motions for the kepler problem

$x(t) = r(t)\,\underline{v}$, $\quad \ddot{r}(t) = -\frac{\mu}{[r(t)]^2} = V'(r(t))$

with $V(s) = \frac{\mu}{s}$ and $h = \frac{1}{2}\dot{r}^2 - \frac{\mu}{r}$ conserved.

$\Rightarrow -V(r) \le h \qquad$ i.e. $-\frac{\mu}{r} \le h$

$h \geq 0$: the condition is always satisfied
$h < 0$: it corresponds to $r \leq -\frac{\mu}{h}$



from the energy relation:

$$\dot{r}(t) = \pm\sqrt{\frac{2\mu}{r(t)} + 2h}$$

(for any h, two branches)

Rk. $h<0$ corresponds to (i)
$h \geq 0$ " to (ii) ($\dot{r}>0$) or (iii) ($\dot{r}<0$)

Rk. When $h<0$, $r \leq -\frac{\mu}{h}$ (with = when $\dot{r}=0$)

The time to go from $r=0$ to $r=-\frac{\mu}{h}$ is:

$$\dot{r}(t) = \frac{dr}{dt} = \sqrt{\frac{2\mu}{r(t)} + 2h}, \quad \int_{\alpha}^{t_0} dt = \underbrace{\int_0^{-\mu/h} \frac{dr}{\sqrt{2\mu/r + 2h}}}$$

this integral is finite but it is an elliptic integral

Rk. If $h=0$: $t - \alpha = \int_0^{r(t)} \frac{1}{\sqrt{2\mu}} s^{1/2}\, ds$

$$= \frac{1}{\sqrt{2\mu}} \frac{2}{3} r^{3/2} \Rightarrow \boxed{r(t) = C_1 (t-\alpha)^{2/3}}$$