Basics in Celestial Mechanics - L2

Kepler's laws and equation

Kepler's laws

Prop. Any conic with a focal point at $\underline{0}$ is a set of points $\underline{x} \in \mathbb{R}^2$ s.t.

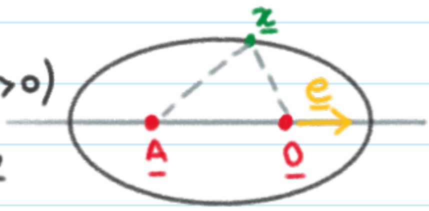
$$\boxed{|\underline{x}| + \langle \underline{e}, \underline{x} \rangle = k} \qquad (C)$$

for some $\underline{e} \in \mathbb{R}^2$ and $k \in \mathbb{R}$.
Furthermore, an equation of the form (C) is a conic with a focal point at $\underline{0}$ when

$\varepsilon = |e|$

- $|e| < 1$ and $k > 0$: ellipse
- $|e| = 1$ and $k > 0$: parabola
- $|e| > 1$ and $k > 0$: branch of iperbola (closer to $\underline{0}$)
- $\quad$ " $\quad$ and $k < 0$: branch of iperbola (far from $\underline{0}$)

Proof. Ellipse $\mathcal{E}$. Focal points : $\underline{A}, \underline{0}$



$$\underline{x} \in \mathcal{E} \iff |\underline{x}| + |\underline{A} - \underline{x}| = c \ (>0)$$

$$\iff |\underline{A} - \underline{x}| = c - |\underline{x}|$$

$$\underset{(\text{both } \geq 0)}{\iff} |\underline{A} - \underline{x}|^2 = (c - |\underline{x}|)^2$$

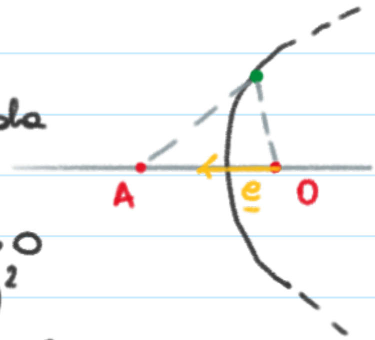
$$\iff |\underline{x}| - \frac{1}{c}\langle \underline{A}, \underline{x} \rangle = \frac{c^2 - |\underline{A}|^2}{2}$$

$$\underline{e} := -\frac{1}{c}\underline{A} \ , \quad k := \frac{c^2 - |\underline{A}|^2}{2}$$

Rk. $|\underline{e}| < 1$ and $k > 0$, indeed $|\underline{A}| < |\underline{x}| + |\underline{A} - \underline{x}| = c$

if $=$, the ellipse reduces to two points

Hyperbola $\mathcal{H}$. Focal points $\underline{A}, \underline{O}$.



$\mathcal{H}_0$ = branch of hyperbola closer to $\underline{O}$

$$x \in \mathcal{H}_0 \iff |\underline{x}-\underline{A}| - |\underline{x}| = c > 0$$
$$\iff |\underline{x}-\underline{A}|^2 = (|\underline{x}|+c)^2$$
$$\iff |\underline{x}| + \frac{1}{c}\langle \underline{A}, \underline{x}\rangle = \frac{|\underline{A}|^2 - c^2}{2c}$$

$$\underline{e} := \frac{1}{c}\underline{A} \qquad k := \frac{|\underline{A}|^2 - c^2}{2c}$$

Rk. $|\underline{e}| > 1$ and $k > 0$ since $|\underline{A}| > c$
$(|A| > |A-x| - |x| = c)$

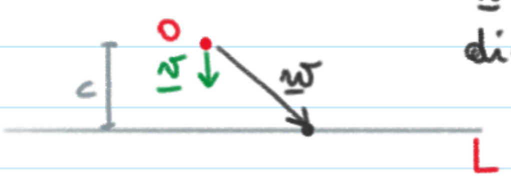
Rk. $\mathcal{H}_A$ = branch of $\mathcal{H}$ closer to $\underline{A}$

$$x \in \mathcal{H}_A \iff |\underline{x}| - |\underline{x}-\underline{A}| = c > 0$$
$$\iff \dots \quad |\underline{x}| - \frac{1}{c}\langle \underline{A}, \underline{x}\rangle = \frac{c^2 - |\underline{A}|^2}{2c}$$

Parabola $\underline{P}$. Focal point O.

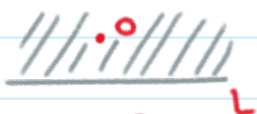
$\underline{\nu}$: $|\nu| = 1$

directrix L: $\underline{w} \in \mathbb{R}^2 : \langle \nu, \underline{w}\rangle = c$



$$\underline{x} \in \underline{P} \iff d(\underline{x}, L) = |\underline{x}|$$

since $d(x,L) = \begin{cases} c - \langle \underline{x}, \underline{v}\rangle, & \text{if } \underline{x} \in \\ \langle \underline{x}, \underline{v}\rangle - c, & \text{if } \underline{x} \in \end{cases}$



then $\underline{x} \in P$ iff it belongs to the half plane containing $\underline{O}$ and it satisfies $c - \langle \underline{x}, \underline{v}\rangle = |\underline{x}|$

$$\underline{e} = \underline{\nu} \qquad k = c$$

Rk. $|\underline{e}| = 1$ and $k > 0$. ∎

1st kepler law: Let $\underline{x} = \underline{x}(t)$ be a solution of

(K) $\quad \ddot{\underline{x}}(t) = -\frac{\mu}{|\underline{x}(t)|^3}\underline{x}(t)$

with angular momentum $\underline{c} \neq \underline{0}$.

Then $\underline{x}(t)$ mooves on a conic with focal point in $\underline{0}$ (and $k>0$ in (c)).

Proof. $\frac{d}{dt}\left(\frac{x}{|x|}\right) = \frac{\dot{x}|x| - x\langle \frac{x}{|x|}, \dot{x}\rangle}{|x|^2}$

$(u \wedge v)\wedge w = \langle u, w\rangle v - \langle v, w\rangle u$ with $u = w = x$, $v = \dot{x}$

$$= \frac{\dot{x}\langle x, x\rangle - x\langle x, \dot{x}\rangle}{|x|^3}$$

$$= \frac{(x\wedge\dot{x})\wedge x}{|x|^3}$$

$$= \underline{c}\wedge\left(-\frac{1}{\mu}\ddot{x}\right) = -\frac{1}{\mu}\frac{d}{dt}(\underline{c}\wedge\dot{x})$$

$\exists v \in \mathbb{R}^2 : \frac{x}{|x|} + \frac{1}{\mu}\underline{c}\wedge\dot{x} = \underline{v}$ (constant vector)

hence $\mu\left(\frac{x}{|x|} - \underline{v}\right) = -\underline{c}\wedge\dot{x}$ (*)

projecting on $\underline{x}$: $\mu\left\langle \frac{x}{|x|} - \underline{v}, x\right\rangle = -\langle \underline{c}\wedge\dot{x}, x\rangle$

$\langle u\wedge v, w\rangle = \langle u, v\wedge w\rangle$

$$\mu(|x| + \langle -v, x\rangle) = -\langle \underline{c}, \underbrace{\dot{x}\wedge x}_{=-\underline{c}}\rangle$$

hence $|x| + \langle -\underline{v}, x\rangle = \frac{|\underline{c}|^2}{\mu}$

which is (c) with $\underline{e} = -\underline{v}$ and $k = \frac{|\underline{c}|^2}{\mu} > 0$.

RK. If $|e| > 1$ (hyperbola) the $\underline{x}$ mooves on the branch closer to $\underline{0}$.

Classification of the keplerian orbit w.r.t. the total energy h

$$\frac{1}{2}|\dot{x}|^2 - \frac{\mu}{|x|} = h \iff |\dot{x}|^2 = \frac{2\mu}{|x|} + 2h$$

from $(*)$: $\mu^2 \left| \frac{x}{|x|} + \underline{e} \right|^2 = |\underline{c} \wedge \dot{x}|^2$

hence $\mu^2 \left(|x|^2 + |\underline{e}|^2 + \frac{2}{|x|} \underbrace{\langle x, \underline{e} \rangle}_{\frac{|c|^2}{\mu} - |x|} \right) = |\underline{c}|^2 |\dot{x}|^2$

replacing:

we obtain: $h = \frac{\mu^2}{2|\underline{c}|^2}(|\underline{e}|^2 - 1)$

Conclusion.

$|e| < 1 \iff h < 0$ ellipse

$|e| = 1 \iff h = 0$ parabola

$|e| > 1 \iff h > 0$ branch of hyperbola

Theorem (global existence)

Let $x : I \subseteq \mathbb{R} \to \mathbb{R}^2$ solves (K) with $\underline{c} \neq \underline{0}$.
Then $I \equiv \mathbb{R}$.

Proof. (K) can be written as $\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -\frac{\mu}{|x(t)|^3} x(t) \end{cases}$

From the 1st K. law, <u>x(t) is bounded from the origin</u> i.e.

$\exists \rho > 0 : |x(t)| > \rho, \forall t$

Then $\begin{cases} |\dot{x}(t)| \leq |y(t)| \\ |\dot{y}(t)| \leq \frac{\mu}{\rho^3} |x(t)| \end{cases}$

since $F(x,y) = \left(y, -\frac{\mu}{|x|^3} x\right)$ is <u>sublinear</u> and C^1

the solution x can be extended on any compact subset of $\mathbb{R}$, hence $I \equiv \mathbb{R}$

[Pagani - Salsa, Analisi Matematica vol. 2

CAP. 4, Teo 1.6] ∎

Theorem (entire conic)

Let x be a solution of (K) with $\underline{c} \neq \underline{0}$.
Then $x(t)$ covers the entire conic.

Proof (when $|e|<1$, for the other cases see [Ortega])

In polar coordinates $x(t) = r(t)(\cos \vartheta(t), \sin \vartheta(t))$
and we can assume

$$\dot{\vartheta}(t) > 0, \forall t \quad \text{so that} \quad |\underline{c}| = r^2(t)\dot{\vartheta}(t)$$

Hence:

$$\dot{\vartheta}(t) = \frac{|\underline{c}|}{r^2(t)} \geq \frac{|\underline{c}|}{R^2}$$

the ellipse is bounded, assume $\mathcal{E} \subseteq B_R(0)$, $R>0$

Hence, since $t \in \mathbb{R}$ we have that

$$\lim_{t \to \pm\infty} \vartheta(t) = \pm\infty$$

, i.e. ϑ is surjective.
[....]

We now find a parametrization of the ellipse with the angle ϑ:

$$e = \varepsilon(\cos\omega, \sin\omega)$$ for some $\varepsilon \in (0,1)$ and $\omega \in [0, 2\pi)$

$$\langle x(t), e \rangle = \varepsilon r(t) \cos(\vartheta(t) - \omega)$$

and (C) reads

$$r(t) = \frac{k}{1 + \varepsilon\cos(\vartheta(t) - \omega)}$$

$\neq 0$ since $\varepsilon \in (0,1)$

The map
$$\begin{cases} \gamma : \mathbb{R} \longrightarrow \mathbb{R}^2 \\ \vartheta \longmapsto \dfrac{k}{1+\varepsilon\cos(\vartheta(t) - \omega)}(\cos\vartheta(t), \sin\vartheta(t)) \end{cases}$$

parametrizes the whole ellipse. ∎

Rk. Since γ is 2π periodic, $x(t)$ passes an infinite number of time through any point of the ellipse.

3rd kepler law. Let $x(t)$ be a solution of (k) with $\underline{c} \neq \underline{0}$ and $h<0$ (ellipse). Then $x(t)$ is periodic with period

$$T = \frac{2\pi}{\mu} a^{2/3}$$

where a = major semiaxes of the ellipse.

Proof. We have shown that

$$x(t) = \frac{k}{1+\varepsilon\cos(\vartheta(t)-\omega)}\left(\cos\vartheta(t), \sin\vartheta(t)\right)$$

and $\dot{\vartheta}(t) \geq C > 0$.

Hence $\vartheta(t)$ is strictly monothone and

$$\exists!\, T>0 \;:\; \vartheta(T) = \vartheta(0) + 2\pi$$

- Such T is a period for $x(t)$ (trivial) and also for $\dot{x}(t)$ indeed:

$$\dot{x}(t) = \underbrace{\dot{r}(t)}_{\text{periodic}}\, \underline{e}_r + \underbrace{r(t)\dot{\vartheta}(t)}_{\frac{r^2(t)\dot{\vartheta}(t)}{r(t)}}\, \underline{e}_\vartheta$$

$r^2(t)\dot{\vartheta}(t) \leftarrow$ const.

- Such T is the minimal period: if $\tilde{T}$ is s.t. $x(\tilde{T}) = x(0)$ then

$$\vartheta(\tilde{T}) = \vartheta(0) + 2N\pi \quad \text{for some } N \geq 1$$

then $\vartheta(\tilde{T}) \geq \vartheta(T) = \vartheta(0) + 2\pi$

hence, since ϑ is monothone,

$$\tilde{T} \geq T$$

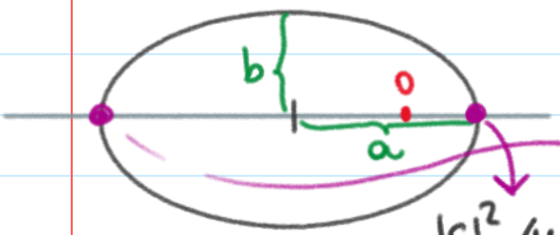
Let us now compute T

t

$$\text{Area}(\mathcal{E}) = \pi a b = \frac{1}{2}\int_0^T r^2(t)\dot{\vartheta}(t)\,dt = \frac{|c|}{2}T$$

$$\Rightarrow T = \frac{2\pi\, ab}{|c|} = \frac{2\pi a^2\sqrt{1-\varepsilon^2}}{|c|}$$ in terms of a?

$$b = a\sqrt{1-\varepsilon^2}$$



$$r(t) = \frac{|c|^2/\mu}{1+\varepsilon\cos(\vartheta(t)-\omega)}$$

$$\frac{|c|^2/\mu}{1+\varepsilon} \le r(t) \le \frac{|c|^2/\mu}{1-\varepsilon}$$

$$\Rightarrow a = \frac{1}{2}\frac{|c|^2}{\mu}\left(\frac{1}{1+\varepsilon}+\frac{1}{1-\varepsilon}\right)$$

$$a = \frac{|c|^2}{\mu(1-\varepsilon^2)}$$

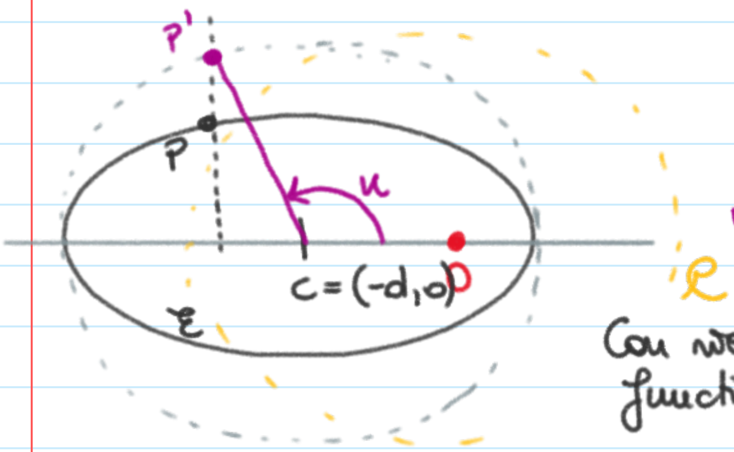
hence

$$a\mu = \frac{|c|^2}{1-\varepsilon^2} \Rightarrow \frac{\sqrt{1-\varepsilon^2}}{|c|} = \frac{1}{\sqrt{a\mu}}$$

Replacing we have : $T = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$ ■

We know that any solution of (k) moves on a conic. Fixed an initial condition, can we determine the position of the partical at t ?

We focus on $h<0$ and we look for a different parametrization of the ellipse.



u = eccentric anomaly

Can we write $P = x(t)$ as a function of u

$$\begin{cases} T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x,y) \mapsto (x+d, y) \end{cases}$$ translation

$$\begin{cases} L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x,y) \mapsto (x, \frac{a}{b} y) \end{cases}$$ endomorphism

$L \circ T : \mathcal{E} \longrightarrow \mathcal{R}$ bijection

$\hookrightarrow \mathcal{R} = \partial B_a(0)$

$$(L \circ T)^{-1} : \mathcal{R} \longrightarrow \mathcal{E}$$
$$(x,y) \longmapsto (x-d, \frac{b}{a} y)$$

$(x,y) \in \mathcal{R} \iff x = a\cos u, \; y = a \sin u$

$$(L \circ T)^{-1}(a\cos u, a \sin u) = (\underset{\text{m.s.}}{a} \cos u - \underset{\varepsilon a}{d}, \underset{a\sqrt{1-\varepsilon^2}}{b} \sin u)$$

$$\Rightarrow \Gamma(u) = a\left(\cos u - \varepsilon, \sqrt{1-\varepsilon^2} \sin u\right)$$

$$\underline{x}(t) = a\left(\cos[u(t)] - \varepsilon, \sqrt{1-\varepsilon^2} \sin[u(t)]\right)$$

Claim: write $u = u(t)$

We use once more the conservation of the angular momentum: $\underline{c} = x_1 \dot{x}_2 - \dot{x}_1 x_2$ so that:

$$\dot{x}(t) = a\left(-\sin[u(t)], \sqrt{} \cos[u(t)]\right)\dot{u}(t)$$

$$\dot{u}(t) = \frac{|c|}{a^2\sqrt{1-\varepsilon^2}\underbrace{\left(1-\varepsilon\cos[u(t)]\right)}_{\neq 0, \; > 0}}$$ 1^{st} order o.d.e.

$$\int \left[1 - \varepsilon \cos[u(t)]\right] \dot{u}(t) \, dt = \int \frac{|c|}{a^2\sqrt{1-\varepsilon^2}} \, dt$$

$$u(t) - \varepsilon \sin[u(t)] = \frac{|c|}{\underbrace{a^2\sqrt{1-\varepsilon^2}}_{\sqrt{1-\varepsilon^2} = \frac{|c|}{\sqrt{\mu a}}}}\, t = \frac{\sqrt{\mu}}{a^{3/2}}\, t$$

$$\boxed{u - \varepsilon \sin u = \frac{\sqrt{\mu}}{a^{3/2}}\, t}$$

we have found $t = t(u)$ the inverse function!

Eq. di keplero : permette di dire a quale istante un pianeta si trova nel punto di anomalia u

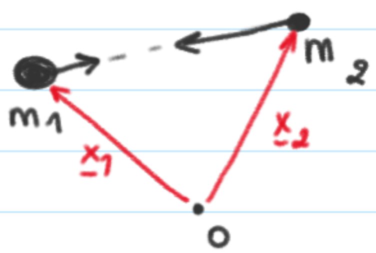
series solutions
$\rightsquigarrow$ Bessel functions

2-body problem

$\underline{x}_1(t)$, $\underline{x}_2(t)$

m_1 m_2 (>0)

$$(2B) \begin{cases} \ddot{\underline{x}}_1 = G m_2 \dfrac{\underline{x}_2 - \underline{x}_1}{|\underline{x}_2 - \underline{x}_1|^3} \\ \ddot{\underline{x}}_2 = G m_1 \dfrac{\underline{x}_1 - \underline{x}_2}{|\underline{x}_2 - \underline{x}_1|^3} \end{cases}$$



$\Delta = \{(x_1, x_2) \in \mathbb{R}^6 \ : \ x_1 = x_2\}$ singular set

$$\underbrace{m_1 \ddot{\underline{x}}_1 + m_2 \ddot{\underline{x}}_2}_{\frac{d^2}{dt^2}(m_1 \underline{x}_1 + m_2 \underline{x}_2)} = \underline{0}$$

$$\frac{d^2}{dt^2}\big(\underbrace{m_1 \underline{x}_1 + m_2 \underline{x}_2}_{\text{c. of mass}}\big) = 0$$

$\| \rightarrow$ the motion of the center of mass in inertial

$\Rightarrow$ we suppose it fixed at the origin

$\mapsto$ $m_1 x_1 + m_2 x_2 = 0$ so that $\boxed{x_2(t) = -\frac{m_1}{m_2} x_1(t)}$

The vector $\underline{x}_1(t) - \underline{x}_2(t)$ can be written in terms of $\underline{x}_1$ or $\underline{x}_2$:

$$\underline{x}_1(t) - \underline{x}_2(t) = \frac{m_1+m_2}{m_2}\, \underline{x}_1(t)$$

$$= \frac{m_1+m_2}{m_1}\, \underline{x}_2(t)$$

and system (28) reads:

$$\begin{cases} \ddot{\underline{x}}_1(t) = -G\mu_1 \dfrac{\underline{x}_1(t)}{|\underline{x}_1(t)|^3} \\ \ddot{\underline{x}}_2(t) = -G\mu_2 \dfrac{\underline{x}_2(t)}{|\underline{x}_2(t)|^3} \end{cases}$$

2 uncoupled Kepler equations!

with $\mu_1 = \frac{m_2^3}{(m_1+m_2)^2}$ $\mu_2 = \frac{m_1^3}{(m_1+m_2)^2}$

Conclusions: $x_1(t)$ and $x_2(t)$ move on conics linked by the relation $\underline{x}_2(t) = -\frac{m_1}{m_2}\underline{x}_1(t)$

hence

$$|\underline{x}_1(t)| + \langle \underline{e}, \underline{x}_1(t)\rangle = k$$

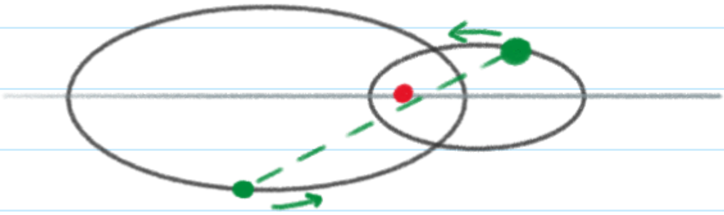
and

$$|\underline{x}_2(t)| - \langle \underline{e}, \underline{x}_2(t)\rangle = \lambda k$$

for some fixed vector $\underline{e}$ and $\lambda = \frac{m_1}{m_2}$.

In particular the motions of the two bodies are coplanar.

Ex.



if $|e|<1$