Basics in Celestial Mechanics - L3

The N-body problem

$$(1)\quad M\ddot{\underline{x}}(t) = \nabla V(\underline{x}(t)) \longrightarrow (S)\begin{cases}\dot{\underline{x}}(t) = \underline{v}(t)\\ \dot{\underline{v}}(t) = M^{-1}\nabla V(\underline{x}(t))\end{cases}$$

- $M = \mathrm{diag}(m_1 \,—\, m_N)$
- $\underline{x}(t) = (\underline{x}_1(t), \,—\, \underline{x}_N(t)) \in (\mathbb{R}^3)^N$
- $V(x) = \sum_{1\le i<j\le N} \dfrac{m_i m_j}{|\underline{x}_i - \underline{x}_j|^\alpha} \qquad \alpha \in (0,2)$

$$\Delta = \{\underline{x} = (\underline{x}_1 — \underline{x}_N) \in (\mathbb{R}^3)^N : \underline{x}_i = \underline{x}_j \text{ for some } i \neq j\}$$

↳ singular set (cone, closed)

$$V \in C^1(\Omega),\quad \Omega = \mathbb{R}^{3N} \setminus \Delta$$

$$(1)\begin{cases}\ddot{\underline{x}}_i(t) = \sum_{j\neq i} m_i m_j \underbrace{\dfrac{\underline{x}_j(t) - \underline{x}_i(t)}{|\underline{x}_j(t) - \underline{x}_i(t)|^{\alpha+2}}}_{F_{ji}(t)}\\ i = 1, \dots, N\end{cases}$$

→ $\sum_{i=1}^{N} \ddot{\underline{x}}_i(t) = 0 \Rightarrow$ the center of mass can be fixed at 0

→ $h = \underbrace{\frac{1}{2}\langle M\dot{\underline{x}}, \dot{\underline{x}}\rangle}_{k(\dot{\underline{x}})} - V(\underline{x})$ is conserved (energy)

→ $\underline{c} = \sum_{i=1}^{N} m_i\, \underline{x}_i(t) \wedge \dot{\underline{x}}_i(t)$ is conserved (total angular momentum).

→ $\underline{x}$ solves (1), $A \in SO(3) \Rightarrow A\underline{x}$ solves (1)

Rk. If $N=2$, system (1) can be reduced to a central force system.

If $N>2$ the force acting on each body is

$$F_i(\underline{x}(t)) = \sum_{j\neq i} m_i m_j \frac{\underline{x}_j(t) - \underline{x}_i(t)}{|\underline{x}_j(t) - \underline{x}_i(t)|^{\alpha+2}}$$

generated by

$$V_i(\underline{x}(t)) = \frac{1}{2}\sum_{\substack{j=1\\ j\neq i}}^{N} m_i m_j \frac{1}{|\underline{x}_j(t) - \underline{x}_i(t)|^\alpha}$$

which is anisotropic, i.e. $V_i(\underline{x}) = \frac{1}{|\underline{x}|^\alpha} V_i\left(\frac{\underline{x}}{|\underline{x}|}\right)$

with $V_i\big|_{S^2}$ non-constant.

"Simple" solutions

① Constant solutions do not exist.

($\bar{\underline{x}}$ const sol. $\Rightarrow \nabla V(\bar{\underline{x}}) = 0$

$\Rightarrow \nabla V(\bar{\underline{x}}) \cdot \bar{\underline{x}} = -\alpha V(\bar{\underline{x}}) = 0$ ↯)

Euler's Theorem for homog. functions

$\Omega \subseteq \mathbb{R}^d$ open cone

$f : \Omega \to \mathbb{R}$, $f(\lambda \underline{x}) = \lambda^p f(\underline{x})$ for some $p \in \mathbb{R}$

$\forall \lambda > 0, \forall \underline{x} \in \Omega$

$\Rightarrow \nabla f(\underline{x}) \underline{x} = p f(\underline{x})$, $\forall \underline{x} \in \Omega$.

② Self similar solutions (omographic)

Does (1) admit solutions

of the form : $\boxed{\underline{x}(t) = \lambda(t) A(t) \bar{\underline{x}}}$ $t \in I \subseteq \mathbb{R}$

for some : $\bar{\underline{x}} \in \mathbb{R}^{3N}$ (constant shape)

$\lambda : I \to \mathbb{R}^+$ dilation

$A : I \to SO(3)$ isometry

(OM)

Theorem (Wintner § 371-375)

Let $\underline{x}$ be a sol. of (1) on I satisfying (OM). Then:

(i) $\underline{x}$ is planar

OR

(ii) $A(t) \equiv I_3$

$\boxed{\text{Case (ii)}}$ homothetic solutions $\underline{x}(t) = \lambda(t) \bar{\underline{x}}$,

for which $\lambda(t)$ and $\bar{\underline{x}}$?

Replacing in $M\ddot{\underline{x}} = \nabla V(\underline{x})$ we obtain:

$$(\cdot) \quad M \ddot{\lambda}(t) \bar{\underline{x}} = \nabla V(\lambda(t) \bar{\underline{x}}) = \frac{1}{\lambda^{\alpha+1}(t)} \nabla V(\bar{\underline{x}})$$

mult. by $\bar{x}$: $\quad \ddot{\lambda}(t)\underbrace{\langle M\bar{x},\bar{x}\rangle}_{\substack{2I(\bar{x})\\ \text{inertial mom.}}} = \frac{1}{\lambda^{\alpha+1}(t)}\underbrace{\langle \nabla V(\bar{x}),\bar{x}\rangle}_{-\alpha V(\bar{x})}$

$$(\lambda) \quad \boxed{\ddot{\lambda}(t) = -\frac{\alpha V(\bar{x})}{2I(\bar{x})}\frac{1}{\lambda^{\alpha+1}(t)}}$$

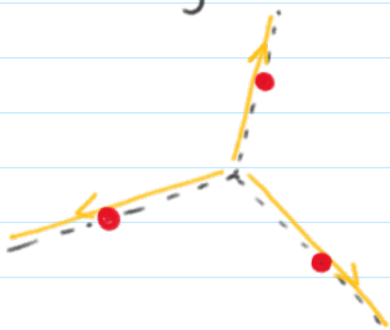
$\Rightarrow$ λ solves a 1-dimentional α-kepler pb

while (replacing (λ) in $(\cdot)$) we get:

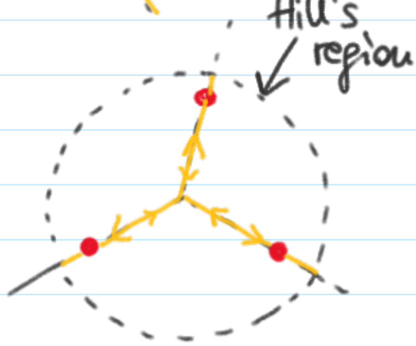
$$(\bar{x}) \quad \boxed{M\bar{x} = -\frac{2I(\bar{x})}{\alpha V(\bar{x})}\nabla V(\bar{x})}$$

Fixed a "special shape" $\bar{x}$ each body mooves on a straight line connecting it to the center of mass:

- if $h \geq 0$: unbounded motion from/to the origin
  $(\alpha,+\infty)$ $(-\infty,\omega)$



- if $h<0$: ejection-collision unbounded motion



Rk. If $h=0$, we can solve explicitely the 1-dim Kepler problem $\boxed{\lambda(t) = C_1(t-\alpha)^{\frac{2}{2+\alpha}}}$ if $t\in(\alpha,+\infty)$, or $c(\omega-t)^{2/2+\alpha}$ if $t\in(-\infty,\omega)$.

$\boxed{\text{Case (i)}}$ homographic solution $\underline{x}(t) = \phi(t)\,\bar{\underline{x}}$

where $\begin{cases} \phi : \mathbb{R} \to \mathbb{C} \\ \quad t \mapsto \lambda(t) \underbrace{e^{i u(t)}}_{A(t)} \end{cases}$

for which $\bar{\underline{x}}$ and $\phi(t)$?

- ϕ solves the 2-dim α-kepler problem
- $\bar{x}$ solves $(\bar{\underline{x}})$

Rk. If $\alpha = 1$, is a conic section

Rk. If $\lambda(t) = 1$ $\forall t$, just rotation, the motion is a relative equilibrium

Solutions of $(\bar{\underline{x}})$, or central configurations

$\bar{x} \in \mathbb{R}^{3N}$, solution of $(\bar{x})$ can be seen as critical points of $V|_{I=1}$.

Indeed: if $\bar{x}$ is a critical point of $V|_{I=1}$ then (Lagrange multiplier):

$$\begin{cases} \nabla V(\bar{\underline{x}}) = \ell \nabla I(\bar{\underline{x}}) & \text{for some } \ell \in \mathbb{R} \\ I(\underline{x}) = 1 \end{cases}$$

$\nabla I(\bar{\underline{x}}) = M\bar{\underline{x}} \Rightarrow \nabla V(\bar{\underline{x}}) = \ell M \bar{\underline{x}}$

mult. by $\bar{x}$: $-\alpha V(\bar{\underline{x}}) = 2\ell \underbrace{I(\bar{\underline{x}})}_{=1}$

$$\Rightarrow \ell = -\frac{\alpha}{2} V(\bar{\underline{x}}) = -\frac{\alpha}{2} \frac{V(\bar{\underline{x}})}{I(\bar{\underline{x}})}$$

replacing in $\nabla V(\bar{x}) = \ell \nabla I(\bar{x})$ we obtain $(\bar{x})$

Solutions interacting with the singular set Δ

Let $(x_0, v_0) \in \Omega \times (\mathbb{R}^3)^N$

then $(\mathcal{C}) \begin{cases} M\ddot{x} = \nabla V(x) \\ x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$ admits a unique sol. $\tilde{x}(t)$, defined on (α, ω) with $-\infty \leq \alpha < \omega \leq +\infty$

What about $\lim_{t\to\omega^-} \tilde{x}(t)$ $\left(\text{or } \lim_{t\to\alpha^+} \tilde{x}(t)\right)$?

Painlevé's theorem (1897)

If $\tilde{x}$ solves (PC) on (α,ω) with $\omega<+\infty$ then
$$\lim_{t\to\omega^-} V(\tilde{x}(t)) = +\infty,$$ i.e. $\tilde{x}$ has a singularity at ω.

Rk. If $\tilde{x}$ has a sing. at ω then:
$$\liminf_{t\to\omega^-} \operatorname{dist}(\tilde{x}(t),\Delta) = 0$$
$$\parallel$$
$$\liminf_{t\to\omega^-} \underbrace{\min_{i\neq j} |\tilde{x}_i(t)-\tilde{x}_j(t)|}_{\tilde{\rho}(t)} = 0$$

⚠ $\lim_{t\to\omega^-} \tilde{x}(t)$ doesn't necessarely exist

def. We say that $\tilde{x}$ has a collision at ω if
$$\lim_{t\to\omega^-} \tilde{x}(t) = \xi \in \Delta$$

- if $\xi = \underline{0}$, $\tilde{x}$ has a total collision at ω
- otherwise at ω $\tilde{x}$ has one (or more) partial collision

- It could be: $\lim_{t\to\omega^-} \|\tilde{x}(t)\| = +\infty$ [Painlevé - Xia]

Von Zeipel's theorem (1908)

Let $\tilde{x}$ be a solution on (α,ω), $\omega<+\infty$
If $\lim_{t\to\omega^-} \|\tilde{x}(t)\| < +\infty$ then $\lim_{t\to\omega^-} \tilde{x}(t) = \xi$.

Lagrange-Jacobi equation

x sol. of (1)

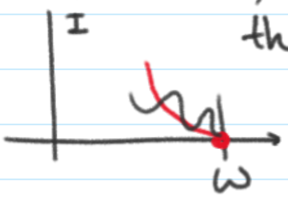
$I(t) = \frac{1}{2}\langle Mx, x\rangle$ moment of inertia

$\dot{I}(t) = \langle Mx, \dot{x}\rangle$

$\ddot{I}(t) = 2k(\dot{x}) + \langle Mx, \ddot{x}\rangle = 2(h+V(x)) + \langle \nabla V(x), x\rangle$

$$\boxed{\ddot{I}(t) = 2h + (2-\alpha)V(x) = (2-\alpha)k(\dot{x}) + \alpha h} \quad \text{(LJ)}$$

- if $x(t) \to$ "Δ", then $\ddot{I} \longrightarrow +\infty$
- if $x(t) \to 0$, i.e. $I(t) \to 0$,



then $I(t)$ is strictly convex

$\leadsto$ when $\tilde{x}(t) \xrightarrow{t \to \omega^-} 0$ then $|\tilde{x}(t)|$ is monothone decresing and strictly convex

$\to$ total collisions are isolated

Sundman's theorem. Let $\bar{x}$ be a sol. of (1) on (α, ω) with $\omega < +\infty$ and $\lim_{t \to \omega^-} \bar{x}(t) = 0$

Then the (total) angular momentum of $\bar{x}$ is 0.

Pf. Step 1 $c = \sum_i m_i\, x_i \wedge \dot{x}_i$

$$|c| \leq \sum_i m_i\, |x_i \wedge \dot{x}_i| \leq \sum_i m_i |x_i||\dot{x}_i|$$

$$= \sum_i \sqrt{m_i}\, |x_i| \sqrt{m_i}\, |\dot{x}_i| \underset{CS}{\leq} \underbrace{\sqrt{\sum m_i |x_i|^2}}_{\sqrt{2I(x)}} \underbrace{\sqrt{\sum m_i |\dot{x}_i|^2}}_{\sqrt{2k(\dot{x})}}$$

$$|c|^2 \leq 4\, I(x) k(\dot{x})$$

$$\Rightarrow k(\dot{x}) \geq \frac{|c|^2}{4 I(x)}$$

Step 2. From (LJ) $\ddot{I} = \alpha h + (2-\alpha) k(\dot{x}) \geq \alpha h + \frac{2-\alpha}{4 I(\bar{x})} |c|^2$

Since : $I(x) \longrightarrow 0$ as $t \to \omega^-$

and $\ddot{I}(x) \longrightarrow +\infty$

then $\dot{I}(x)$ is strictly increasing (near ω)

and necessarely have a negative sign (since $I \to 0$ and $I > 0$)

Assuming $|c| \neq 0$ we find a contradiction .... ∎

Asymptotic estimates

Let $\bar{x}(t)$ be a sol. for (1) on (α, ω) with $\omega < +\infty$ and assume

$$\lim_{t \to \omega^-} \bar{x}(t) = \underline{0} \qquad (c = 0)$$

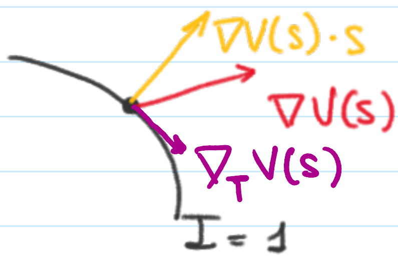
Can we say anything about $r(t) = |\bar{x}(t)|$ and $\underbrace{\dfrac{\bar{x}(t)}{|\bar{x}(t)|}}_{s(t)}$ when $t \to \omega^-$?

$\bar{x}(t) = r(t)\, s(t)$

$$M\ddot{x} = \nabla V(x) \quad \rightsquigarrow \quad \begin{cases} \ddot{r} - r|\dot{s}|^2 = -\dfrac{\alpha}{r^{\alpha+1}} V(s) & (r) \\ \ddot{s} + 2\dfrac{\dot{r}}{r}\dot{s} + |\dot{s}|^2 s = \dfrac{\nabla_T V(s)}{r^{2+\alpha}} & (s) \end{cases}$$

with $\nabla_T V(s) = \nabla V(s) - (\nabla V(s) \cdot s) s$

$$h = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 |\dot{s}|^2 - \frac{V(s)}{r^\alpha}$$



$$\Gamma(t) := r^\alpha \left(\frac{1}{2} r^2 |\dot{s}|^2 - \frac{V(s)}{r^\alpha} \right)$$

- Γ is $\nearrow$ and bounded
- $\exists\, b > 0 \ : \ \lim_{t \to \omega^-} \Gamma(t) = b$

Sundman/Sperling's estimates

1. $r(t) \sim C_1 (\omega - t)^{2/2+\alpha}$ as $t \to \omega$
2. $\lim_{t \to \omega^-} V(s(t)) = b$
3. $\lim_{t \to \omega^-} \operatorname{dist}(s(t), \mathcal{E}^b) = 0$ $\longrightarrow$ set of central config. of V at level b

Rks. From 1. : when $t \to \omega$; $|x(t)|$ decreases as the solution of the 1-dim. α-kepler with $h = 0$

$\rightsquigarrow$ a "blow-up" will have a 0-energy trajectory as a limit

From 3. : the "limiting shape" $(s(t))$ approaches the set $\mathcal{E}^b$

- If $\mathcal{E}^b$ contains a continuum of central central configurations, then $s(t)$ could present a "spin" when approaching a collision.
- Very few is known about $\mathcal{E}^b$ when N is greater than 3.

Further developments

- Variational methods (direct, MP, ..) to find solutions
  $\rightsquigarrow$ as. estimates can be used to avoid collisions (weak sol $\to$ strong sol.)
- When a limiting config. $\bar{s}$ exists, can we compute the Morse index of $\bar{s}$ with the Morse Index of a colliding sol. as. to $\bar{s}$?