Basics in Celestial Mechanics - L4

The planar restricted 3-boby problem

3BP : $\begin{cases} m_i \ddot{\underline{x}}_i = \sum_{j=1, j\neq i}^{3} G \dfrac{m_i m_j}{|\underline{x}_j - \underline{x}_i|^3}(\underline{x}_j - \underline{x}_i) \\ i = 1,2,3 \end{cases}$

$\underline{x}_i \in \mathbb{R}^3$

Assume that : $\boxed{m_3 \to 0}$ (the 3rd body is negligible withe respect to the 1st and 2nd)

hence :

$$\left.\begin{aligned} \ddot{\underline{x}}_1 &= G \frac{m_2}{|\underline{x}_2 - \underline{x}_1|^3}(\underline{x}_2 - \underline{x}_1) \\ \ddot{\underline{x}}_2 &= G \frac{m_1}{|\underline{x}_1 - \underline{x}_2|^3}(\underline{x}_1 - \underline{x}_2) \end{aligned}\right\} \text{TWO PRIMARIES FORM A 2-BP SYSTEM}$$

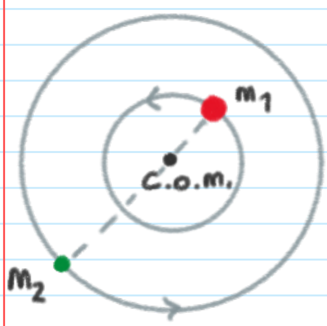
$$\boxed{\ddot{\underline{x}}_3 = G \frac{m_1}{|\underline{x}_1(t) - \underline{x}_3|^3}(\underline{x}_1(t) - \underline{x}_3) + \frac{G}{|\underline{x}_2(t) - \underline{x}_3|^3}(\underline{x}_2(t) - \underline{x}_3)}$$

given $x_1(t)$ and $\underline{x}_2(t)$ this is a second order time dependent ODE

Assume that :

(i) $\underline{x}_1(t), \underline{x}_2(t)$ has circular trajectories with center of mass at $\underline{0}$

$m_1 \underline{x}_1(t) + m_2 \underline{x}_2(t) = \underline{0}$



We normalize assuming:

$G = 1$

$m_1 = 1 - m_2$, $m_2 := \mu \in (0, 1/2]$

$m_1 = m_2$, same radius

$\underline{x}_1(t) = -\mu e^{it}$ $\quad \underline{x}_2(t) = (1-\mu)e^{it}$

(ii) x_3 lies on the same plane of x_1 and x_2

We obtain the RCP3BP

$m_3 \to 0$ (i) (ii)

$$(\underline{x}_3) \quad \begin{cases} \ddot{\underline{x}}_3 = \dfrac{1-\mu}{|-\mu e^{it} - \underline{x}_3|^3}\left(-\mu e^{it} - \underline{x}_3\right) + \dfrac{\mu}{|(1-\mu)e^{it} - \underline{x}_3|^3}\left((1-\mu)e^{it} - \underline{x}_3\right) \\ \underline{x}_3 \in \mathbb{R}^2 \end{cases}$$

We pass to a rotating coord. system where the primaries are fixed:

$\underline{x}_1(t) \rightsquigarrow P_1 = (-\mu, 0)$

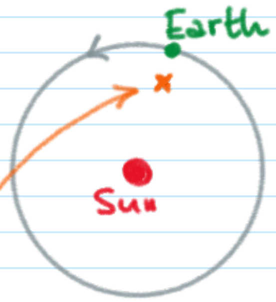
$\underline{x}_2(t) \rightsquigarrow P_2 = (1-\mu, 0)$

$$(*) \quad \boxed{\begin{array}{c} \underline{x}_1(t) = R(t)\begin{pmatrix} -\mu \\ 0 \end{pmatrix} \quad x_2(t) = R(t)\begin{pmatrix} 1-\mu \\ 0 \end{pmatrix} \\ \underline{x}_3(t) = R(t)\,\underline{z}(t) \end{array}}$$

with
$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$ orthogonal matrix

Rk. If $\underline{x}_3$ is a stationary sol. in this system then $x_3(t)$ has a circular motion in the inertial frame.

Rk. If $m_1 >> m_2$ and $\underline{x}_1(t) \sim \underline{0}$ then a stationary sol. in the rotating frame corresponds to a circular motion with the same period of x_2 !!



a satellite in a stationary point in rotating coord. mooves with the Earth

Replacing (*) in (x_3) we obtain:

$$\ddot{x}_3(t) = \frac{d^2}{dt^2}\left(R(t)\underline{z}(t)\right) = \ddot{R}(t)\underline{z}(t) + 2\dot{R}(t)\dot{\underline{z}}(t) + R(t)\ddot{\underline{z}}(t)$$

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad \dot{R}(t) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix} = R(t)k$$

with $\boxed{k = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ symplectic matrix

$$\ddot{R}(t) = -R(t)$$

$$\Rightarrow \ddot{x}_3(t) = R(t)\left[\ddot{z}(t) + 2k\dot{z}(t) - z(t)\right]$$

hence (x_3) reads:

$$\boxed{\underbrace{\ddot{\underline{z}} + 2k\dot{\underline{z}}}_{\text{Coriolis force}} = \underbrace{\underline{z}}_{\text{centrifugal force}} + \frac{1-\mu}{|P_1-\underline{z}|^3}(P_1-\underline{z}) + \frac{\mu}{|P_2-\underline{z}|^3}(P_2-\underline{z})}$$

Defining: $\displaystyle \phi(z) = \frac{1}{2}|\underline{z}|^2 + \frac{1-\mu}{|P_1-\underline{z}|} + \frac{\mu}{|P_2-\underline{z}|}$

we obtain:

$$\boxed{\ddot{z} + 2k\dot{z} = \nabla\phi(z)} \quad (z)$$

The Jacobi (first) integral of the RCP3BP

$z(t)$ sol. of (z) $\quad \boxed{J(t) = 2\phi(z(t)) - |\dot{z}(t)|^2}$ Jacobi first integral

indeed:

$$\dot{J}(t) = 2\langle \nabla\phi(z), \dot{z}\rangle - 2\langle \dot{z}, \ddot{z}\rangle$$
$$= 4\langle \dot{z}, k\dot{z}\rangle = 0$$

J induces the Hill's region for the RCP3BP:

$$2\phi(z(t)) \geq J(t) \equiv J(0) = C$$

$\Rightarrow$ the motion cannot leave the region:

$$\mathcal{H}_c = \left\{ z \in \mathbb{R}^2 : \phi(z(t)) \geq C/2 \right\}$$

Equilibrium points of (2) LAGRANGIAN POINTS (or LIBRATION P.)

$\bar{z} \in \mathbb{R}^2 : \quad \nabla\Phi(\bar{z}) = 0 \iff \bar{z} = \frac{1-\mu}{|\bar{z}-P_1|^3}(\bar{z}-P_1) + \frac{\mu}{|\bar{z}-P_2|^3}(\bar{z}-P_2)$

let $\bar{z} = (x, y) \quad \rho_i = |z - P_i| \quad i = 1, 2$

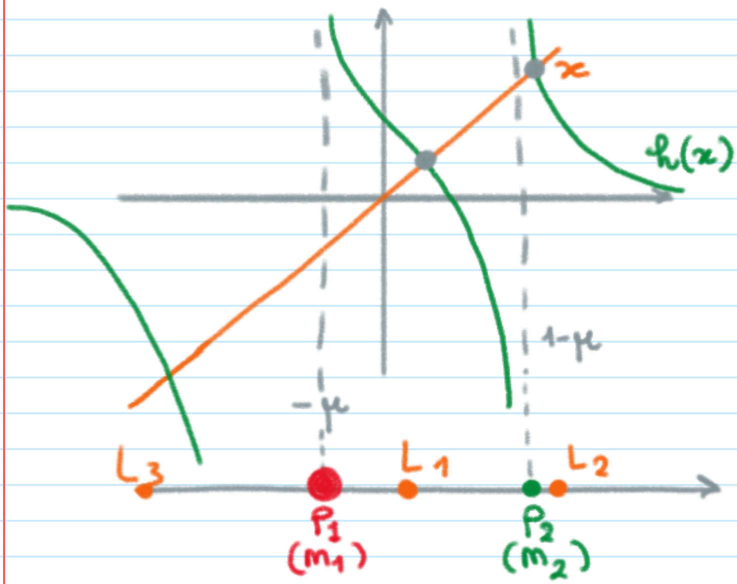
we obtain :

$$\begin{cases} x = \frac{1-\mu}{\rho_1^3}(x+\mu) + \frac{\mu}{\rho_2^3}(x+\mu-1) \\ y = \frac{1-\mu}{\rho_1^3} y + \frac{\mu}{\rho_2^3} y \end{cases}$$

with $x, y \in \mathbb{R}$

$y = 0$ solves the 2^{nd} eq.

1^{st}: $x = h(x)$ with $h(x) = \frac{1-\mu}{\rho_1^3}(x+\mu) + \frac{\mu}{\rho_2^3}(x+\mu-1)$



3 equilibrium points

$y \neq 0 \quad \bar{z} = \frac{1-\mu}{\rho_1^3}(\bar{z}-P_1) + \frac{\mu}{\rho_2^3}(\bar{z}-P_2)$

becomes :

$$\underbrace{\left(1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3}\right)\bar{z}}_{/\!/ \text{ to } \bar{z}} = \underbrace{-\frac{1-\mu}{\rho_1^3}P_1 - \frac{\mu}{\rho_2^3}P_2}_{/\!/ \ x\text{-axes}}$$

since $y \neq 0$ both must vanish

hence : $\begin{cases} 1 - \left(\dfrac{1-\mu}{\rho_1^3} + \dfrac{\mu}{\rho_2^3} \right) = 0 \\ -\dfrac{1-\mu}{\rho_1^3}\mu + \dfrac{\mu}{\rho_2^3}(1-\mu) = 0 \end{cases} \iff \rho_1 = \rho_2 = \rho$

replacing in the 1st eq. : $1 - \dfrac{1-\mu+\mu}{\rho^3} = 0 \iff \rho = 1$

hence $\quad |\bar{z} - P_1| = |\bar{z} - P_2| = 1 \; (= |P_2 - P_1|)$

$\rightsquigarrow$ two more equilib. points.