Basics in Celestial Mechanics - L5

Stability of lagrangian points

We have proved the $\exists$ of 5 equilibria for the equation of the "small" mass in the planar restricted circular 3BP in the rotating frame where the primaries are fixed at $P_1 = (-\mu, 0)$ and $P_2 = (1-\mu; 0)$

$$\boxed{\ddot{z} + 2k\dot{z} = \nabla\Phi(z)} \quad (2)$$

Claim: study local dynamics near L_j, $j = 1 \dots 5$
$\rightsquigarrow$ we linearised (2) at L_j

$$(2) \iff \begin{cases} \dot{z} = w \\ \dot{w} = -2kw + \nabla\Phi(z) \end{cases} \quad (4 \text{ eqs})$$

$$\begin{matrix} z = (x, y) \\ w = (p, q) \end{matrix} \quad \begin{cases} \dot{x} = p \\ \dot{y} = q \\ \dot{p} = 2q + \Phi_x \\ \dot{q} = -2p + \Phi_y \end{cases} \quad (*)$$

this part is linear

If $u = (x, y, p, q)$ then the equilibrium points are $u_j = (x_j, y_j, 0, 0)$ and the linearization of $(*)$ at u_j is:

$$\begin{cases} \dot{X} = P \\ \dot{Y} = Q \\ \dot{P} = 2Q + \Phi_{xx} X + \Phi_{xy} Y \\ \dot{Q} = -2P + \Phi_{yx} X + \Phi_{yy} Y \end{cases}$$

In a matrix notation, put $U = (X, Y, P, Q)$ and obtain:

$\dot{U} = AU$ with

Hessian of Φ at u_j

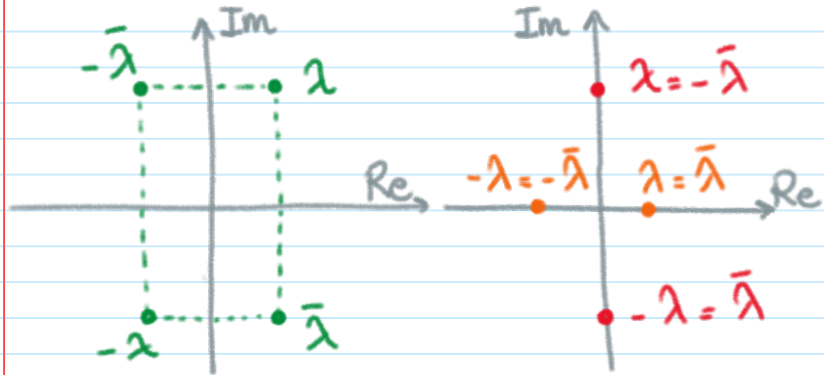
$a_j = \Phi_{xx}(u_j)$ $b_j = \Phi_{xy}(u_j)$

$c_j = \Phi_{yy}(u_j)$ $\longrightarrow$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_j & b_j & 0 & 2 \\ b_j & c_j & -2 & 0 \end{pmatrix}$$

- Eigenvalues of A : $\lambda \,/\, \lambda^4 - (a_j + c_j - 4)\lambda^2 + (a_j c_j - b_j^2) = 0$

Note that : $\lambda \in \mathbb{C}$ solution $\rightarrow -\lambda, \bar{\lambda}, -\bar{\lambda}$ are sols.



- Asymthotic Stability : every eigenv. has $Re < 0$

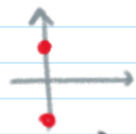
$\Downarrow$

impossible !

- Stability in the past : every eigenv has $Re \geq 0$

  " in the future : " " $Re \leq 0$

In our case : stability in the past. $\Longleftrightarrow$ stability in the future



(indeed $Re\,\lambda \geq 0 \;\forall \lambda \Longleftrightarrow Re\,\lambda = 0 \;\forall \lambda$)

Computation of the eigenvalues of A

$\rightarrow$ we first compute $\Phi_{xx}, \Phi_{xy}, \Phi_{yy}$

$$\nabla \phi(z) = z - (1-\mu)\frac{z - P_1}{|z - P_1|^3} - \mu \frac{z - P_2}{|z - P_2|^3}$$

2x2 matrix

$$\nabla^2 \phi(z) = Id_2 - (1-\mu)\frac{Id_2}{|z - P_1|^3} + 3(1-\mu)\frac{(z - P_1)\otimes(z - P_1)}{|z - P_1|^5} - \mu \frac{Id_2}{|z - P_2|^3} + 3\mu \frac{(z - P_2)\otimes(z - P_2)}{|z - P_2|^5}$$

with $\quad w \otimes w = \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$

hence since $\quad z - P_1 = \begin{pmatrix} x+\mu \\ y \end{pmatrix} \quad z - P_2 = \begin{pmatrix} x+\mu-1 \\ y \end{pmatrix}$

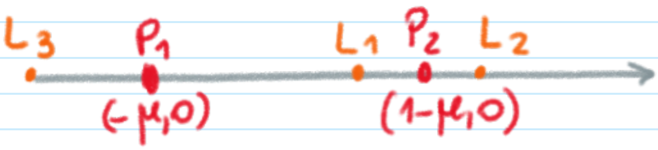
$$\phi_{xx} = 1 - \frac{1-\mu}{\rho_1^3} + 3(1-\mu)\frac{(x+\mu)^2}{\rho_1^5} - \frac{\mu}{\rho_2^3} + 3\mu\frac{(x+\mu-1)^2}{\rho_2^5}$$

$$\phi_{xy} = \phi_{yx} = 3(1-\mu)\frac{(x+\mu)y}{\rho_1^5} + 3\mu\frac{(x+\mu-1)y}{\rho_2^5}$$

$$\phi_{yy} = 1 - \frac{1-\mu}{\rho_1^3} + 3(1-\mu)\frac{y^2}{\rho_1^5} - \frac{\mu}{\rho_2^3} + 3\mu\frac{y^2}{\rho_2^5}$$

When L_1, L_2, L_3 : $y=0$, hence $(x+\mu)^2 = \rho_1^2$

$(x+\mu-1)^2 = \rho_2^2$



$$a_j = \phi_{xx} = 1 + 2\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) > 0$$

$$b_j = 0$$

$$c_j = 1 - \left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_1^3}\right)$$

L_1 : $\rho_1 < 1$, $\rho_2 < 1 \Rightarrow c_1 < 0$

L_2, L_3 : we can compute $x_J c_J = \dots \begin{cases} <0 \text{ at } L_2 \\ >0 \text{ at } L_3 \end{cases}$

hence $c_2 < 0$ and $c_3 < 0$

from :
$$\lambda^4 - (a_J + c_J - 4)\lambda^2 + (a_J c_J - b_J^2) = 0$$

we have $(b_j = 0)$
$$\lambda_\pm^2 = \frac{(a_j + c_j - 4) \pm \sqrt{(a_j + c_j - 4)^2 - 4 a_j c_j}}{2} \quad a_j c_j < 0$$

λ_+^2 is real and >0

⇒ we have 2 real eigenv.
with opposite sign ⇒ L_1, L_2, L_3 are unstable (in Lyapunov sense)

L_4, L_5: $\rho_1 = \rho_2 = 1$ $\quad x+\mu = x+\mu-1 = \frac{1}{2}$

$y = \frac{\sqrt{3}}{2}$

hence $a_J = \frac{3}{4}$, $b_J = \frac{3\sqrt{2}}{4}(2\mu-1)$ $c_J = \frac{9}{4}$

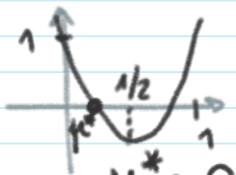
the eq. in λ is: $\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$

$$\lambda_\pm^2 = \frac{-1 \pm \sqrt{1-27\mu(1-\mu)}}{2}$$

- if $\lambda_\pm^2$ are both real and negative ⇒ the linear system is stable
  non linear?

- if $\lambda_+^2 > 0$ ⇒ instability

Let $f(\mu) := 1-27\mu(1-\mu)$



$\mu^* \sim 0.0385$

- if $\mu \in (0, \mu^*]$, then $\lambda_\pm^2$ are real and <0 ⇒ 4 eigen. in $i\mathbb{R}$
  (indeed $0 < f(\mu) < 1$)

- if $\mu \in (\mu^*, 1/2]$, then $\lambda_\pm^2 \in \mathbb{C} \setminus \mathbb{R}$
  ⇒ 4 complex distinct eigenv.
  ⇝ 2 have Re >0
  → instable.

Concrete cases: $m_2 = \mu$, $m_1 = 1-\mu$

$$\frac{m_1}{m_2} = \frac{1}{\mu} - 1$$

linear stab. $\iff \mu < \mu^* \iff \frac{1}{\mu} - 1 > \frac{1}{\mu^*} - 1$
$\qquad$ 21
$\qquad$ 24.6

$\frac{\text{Sun}}{\text{Jupiter}} \sim 10^3$ $\quad \frac{\text{Earth}}{\text{Moon}} \sim 82$

$\rightsquigarrow$ no hope to get it

Non-linear stability. Using "Lyapunov center theorem" we can prove the $\exists$ of periodic solutions near the eq. points for some range of μ
$\rightsquigarrow$ this periodic sols in the rotating frame correspond to quasi-periodic sols in the inertial reference system.
[Ambrosetti - Prodi, A Primer in nonlinear analysis, CAP. 7]