(1) for any $\mu \leq 0$ (2.12) has no periodic trajectories but the trivial one $x = 0,\ y = 0$,

(2) for any $\mu > 0$ (2.12) has a unique periodic solution, which is asymptotically stable. See Figures 7.2 and 7.3.

Note also that the trivial solution $x = 0, y = 0$ is stable for all $\mu < 0$ and unstable for $\mu > 0$.

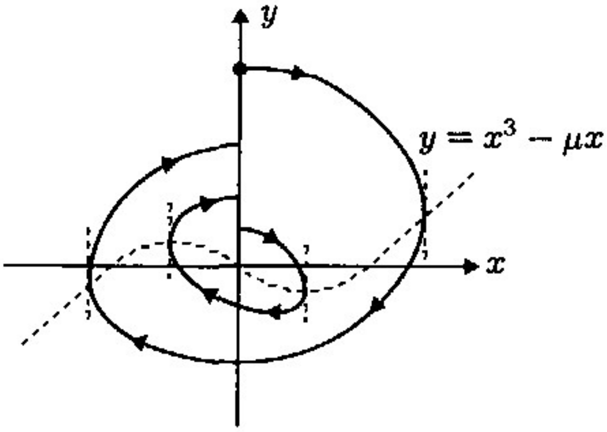

Figure 7.2 Phase portrait of (2.12)

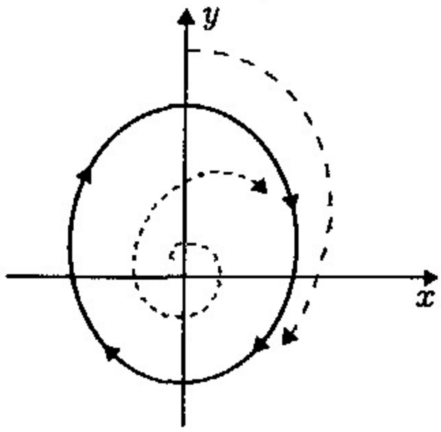

Figure 7.3 The closed orbit of (2.12)

3 The Lyapunov Centre Theorem

Consider the first-order system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), \tag{S}$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$. A *singular point* of (S) is a $p \in \mathbb{R}^n$ such that $f(p) = 0$. In this section we will be concerned with the existence of small oscillations of (S) near an equilibrium p, namely periodic solutions of (S) with orbits confined near p. Let us suppose that $p = 0$ is a singular point of (S) and let $A = f'(0)$. If no point of the spectrum of A belongs to the imaginary axis, then the behaviour of the solutions of (S) near $p = 0$ is completely understood. It is possible to show (see, for example [**P**]) that there are two invariant manifolds M and N, with $\dim(M)+\dim(N) = n$ and $M \cap N = \{0\}$ such that for all $q \in M$ (resp. N) the solution of the Cauchy problem

$$\left.\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= f(u), \\ u(0) &= q, \end{aligned}\right\}$$

tends to 0 as $t \to +\infty$ (as $t \to -\infty$, respectively). The behaviour of the solutions near $p = 0$ is represented in Figure 7.4.

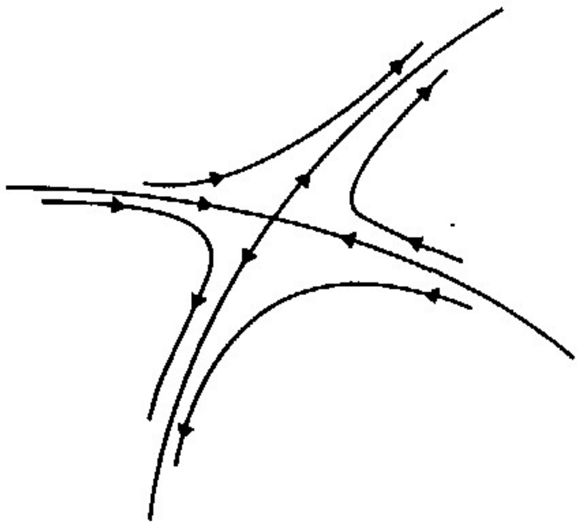

Figure 7.4

As a consequence, a necessary condition for (S) to have closed orbits near a singular point, say $p = 0$, is that A have a pair of purely imaginary eigenvalues $\pm \mathrm{i}\omega_0$. However this condition is not sufficient, in general. For example, consider the two-dimensional system

$$\left.\begin{aligned} x' &= -y - x(x^2 + y^2), \\ y' &= x - y(x^2 + y^2). \end{aligned}\right\} \tag{3.1}$$

The eigenvalues of A are $\pm\mathrm{i}$; on the other hand, if $(x(t), y(t))$ is a solution of (3.1), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}(x^2 + y^2)\right] = xx' + yy' = -(x^2 + y^2)^2.$$

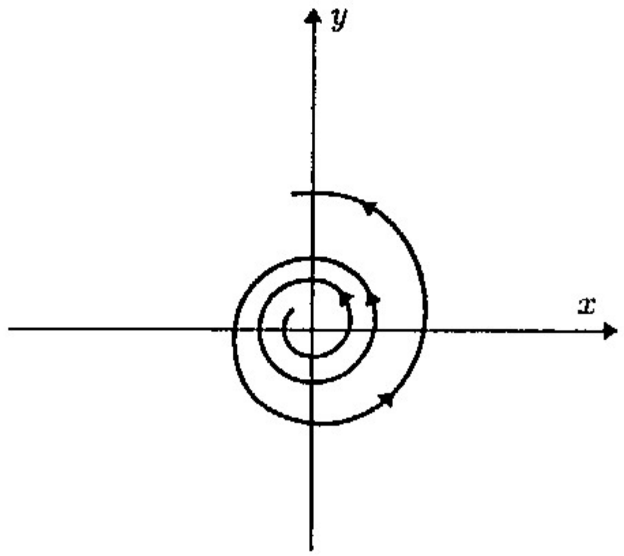

Figure 7.5

It follows that

$$x^2 + y^2 = \frac{1}{2t + c}$$

and hence (3.1) has no periodic solutions (see Figure 7.5).

Lyapunov has shown in a celebrated theorem [**Ly**] that (S) does possess periodic solutions near 0 provided it has a "non-singular" first integral b. Recall that a first integral of (S) is a non-constant real-valued function $b \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $b(u(t)) \equiv$ constant for any solution $u(t)$ of (S). We point out that, in view of the uniqueness of the solutions of the Cauchy problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), \quad u(0) = p,$$

b is a first integral of (S) if and only if

$$f(p) \cdot \nabla b(p) = 0, \text{ for all } p \in \mathbb{R}^n. \tag{3.2}$$

Examples of such systems are the second-order *conservative* (or *gradient*) systems, namely systems like

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \nabla U(u) = 0, \tag{3.3}$$

or the *Hamiltonian systems*

$$\left.\begin{aligned} x' &= -H_y(x,y), \\ y' &= H_x(x,y), \end{aligned}\right\} \tag{HS}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. In fact, the Hamiltonian itself $H = H(x, y)$ is a first integral of (HS). Note that (3.3) is a particular case of (HS): it suffices to take $H(x, y) = \frac{1}{2}|y|^2 + U(x)$.

The following lemma shows the role played by the first integral.

Lemma 3.1 *Suppose b is a first integral of* (S) *and consider the modified system*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u) + \mu\nabla b(u),\ \mu \in \mathbb{R}. \qquad ((\mathrm{S})_\mu)$$

If $u = u(t)$ is a T-periodic solution of (S_μ) *then u is in fact a T-periodic solution of* (S).

Proof. Let $u(t)$ be any solution of (S_μ) for some $\mu \neq 0$. Setting $\beta(t) = b(u(t))$ one has

$$\beta'(t) = \frac{\mathrm{d}}{\mathrm{d}t}b(u(t)) = \nabla b(u(t)) \cdot u'(t) = \nabla b(u(t)) \cdot f(u(t)) + \mu|\nabla b(u(t))|^2$$

Using (3.2) we get that

$$\beta'(t) = \mu|\nabla b(u(t))|^2.$$

If, for example, $\mu > 0$, then $\beta(t)$ is non-decreasing. In the other hand, since x is T-periodic, we deduce $\beta(0) = b(u(0)) = b(u(T)) = \beta(T)$. Hence

$$\beta'(t) = \mu|\nabla b(u(t))|^2 \equiv 0$$

and u solves (S).

Lemma 3.1 suggests we seek small oscillations of a system with a first integral as periodic solutions of (S_μ) above, bifurcating from $\mu_0 = (0,0)$. The following theorem gives conditions under which such a bifurcation occurs.

Theorem 3.2 (Lyapunov Centre Theorem) *Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ is such that $f(0) = 0$. Letting $A = f'(0)$, we suppose that*

(A-1) *A is nonsingular and has a pair of simple eigenvalues $\pm \mathrm{i}\omega_0$;*

(A-2) *for all $k \in \mathbb{Z}$, $k \neq \pm 1$, $\mathrm{i}k\omega_0$ is not an eigenvalue of A.*

Moreover, let us assume that (S) *has a first integral $b \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $b''(0)$ is non-singular.*

Then (S) *possesses small oscillations near $p = 0$.*

More precisely, there exist a neighbourhood J of $s = 0$, a function $\omega(s) \in C^1(J)$, and a family u_s of non-constant, periodic solutions of (S) *such that*

(i) *$\omega(s) \to \omega_0$, as $s \to 0$;*
(ii) *u_s has period $T_s = 2\pi/\omega(s)$;*
(iii) *the amplitude of the orbit u_s tends to 0 as $s \to 0$.*

Proof. According to Lemma 3.1, we can replace (S) with (S_μ) which can be studied by means of Theorem 2.6, with $f(\mu, u) = f(u) + \mu\nabla b(u)$. First of all we note that from (3.2), and recalling that b is C^2 here, it follows that

$$f'(\xi)y \cdot \nabla b(\xi) + f(\xi) \cdot \nabla b''(\xi)y = 0, \quad \text{for all } \xi, y \in \mathbf{R}^n. \tag{3.4}$$

Putting $\xi = 0$, one has

$$Ay \cdot \nabla b(0) + f(0) \cdot b''(0)y = 0, \quad \text{for all } y \in \mathbf{R}^n.$$

Since $f(0) = 0$ and A is non-singular, it follows that $\nabla b(0) = 0$. As a consequence, we infer that $f(\mu, 0) = f(0) + \mu\nabla b(0) = 0$. Moreover, setting $B = b''(0)$, one has

$$A_\mu = f_x(\mu, 0) = A + \mu B.$$

We shall apply Theorem 2.6 with $\mu_0 = 0$. Since $A_0 = A_{\mu_0} = A$, (A_0–1–2) follow from (A–1–2). It remains to verify that (A_0-3) holds.

First, let us consider (3.4). Since f is continuously differentiable, $f(0) = 0$ and b'' is continuous, then it is easy to verify that the map $\xi \to f(\xi) \cdot b''(\xi)y$ is differentiable at $\xi = 0$ with derivative $f'(0)[.] \cdot b''(0)y = A[.] \cdot By$. Hence we can differentiate (3.4) at $\xi = 0$, yielding

$$f''(0)[y, z] \cdot \nabla b(0) + Ay \cdot Bz + Az \cdot By = 0 \quad \text{for all } y, z \in \mathbf{R}^n.$$

Since $\nabla b(0) = 0$ it follows that

$$Ay \cdot Bz + Az \cdot By = 0 \quad \text{for all } y, z \in \mathbf{R}^n,$$

that is (note that B is symmetric),

$$A^T B + BA = 0. \tag{3.5}$$

Then, up to a change of coordinates, the matrix A has the form

$$A = \begin{bmatrix} S & 0 \\ 0 & R \end{bmatrix}$$

with

$$S = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix}$$

and R does not contain $\pm i\omega_0$ in its spectrum, because $\pm i\omega_0$ are simple eigenvalues of A. Let us write

$$B = \begin{bmatrix} U & M \\ M^T & C \end{bmatrix}$$

where U (resp. C) is a symmetric 2×2 (resp. $(n-2) \times (n-2)$) matrix.

From (3.5) it follows readily that

$$SU = US \tag{3.6'}$$

and

$$SM = MR. \tag{3.6''}$$

Recalling that $\omega_0 \neq 0$, from (3.6') one deduces with elementary calculations that there is $\delta \in \mathbb{R}$ such that

$$U = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}.$$

From (3.6″) and using the fact that $\omega_0 \neq 0$ and $\pm i\omega_0$ are not eigenvalues of R, one infers that the $2 \times (n-2)$ matrix M is the 0 matrix. [†]

To see this, let $X, Y \in \mathbb{R}^{n-2}$ denote the two rows of M.

$$M = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Then from (3.6″) it follows X, Y satisfy the system:

$$\begin{cases} XR + \omega_0 Y = 0 \\ YR - \omega_0 X = 0 \end{cases}$$

One finds $X = \omega_0^{-1} YR$ and hence $Y(R^2 + \omega_0^2 I) = 0$. Since $\pm i\omega_0$ are not eigenvalues of R, then one infers that $Y = X = 0$.

From the preceding arguments we deduce that B has, with respect to the same basis used for (3.5), the form

$$B = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & \\ 0 & & C \end{bmatrix}$$

where $\delta \neq 0$, because B is non-singular.

Consequently there results

$$A + \mu B = \begin{bmatrix} \mu\delta & -\omega_0 & 0 \\ -\omega_0 & \mu\delta & \\ 0 & & R + \mu C \end{bmatrix},$$

and hence the branch of eigenvalues $\lambda(\mu)$ such that $\lambda(0) = i\omega_0$ is given by

$$\lambda(\mu) = \mu\delta + i\omega_0.$$

This proves that (A_0–3) holds true. An application of Theorem 2.6, jointly with Lemma 3.1, yields the existence of a C^1 function $\omega(s) \to \omega_0$ and of family u_s of non-constant solutions of (S) with period $T_s = 2\pi/\omega(s)$, such that the amplitude of u_s tends to 0 as $s \to 0$.

Remarks 3.3

(i) The above result being local in nature, it would be sufficient to consider in Theorem 3.2 a vector field f and a first integral b defined in a neighbourhood of 0 in $\mathbb{R}^n$.

(ii) If A has several purely imaginary eigenvalues $\pm i\omega_k, \omega_k > 0$, $k =$

[†] More generally, it is possible to show that *if R and S are square matrices having disjoint spectra and if M is a matrix such that $SM = MR$, then $M = 0$.*

$1, \ldots, m$, the non-resonance condition (A2) is always satisfied at $\pm i\omega^*$, where $\omega^* = \max\{\omega_k, 1 \le k \le n\}$.

(iii) It has been proved by J. Moser [**Mo**] that non-resonance conditions (A1–2) can be eliminated at the expense of the existence of a first integral $b \in C^2(\mathbb{R}^n, \mathbb{R})$ such that $b''(0)$ *is positive-definite*. The following example (see [**Mo**]; see also [**MW**]) shows that, in this more general form, if $b''(0)$ is merely nondegenerate, (S) may have no periodic solutions at all. Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ and consider the Hamiltonian system (HS) with Hamiltonian

$$H(x,y) = \frac{1}{2}(x_1^2 - x_2^2 + y_1^2 - y_2^2) + (|x|^2 + |y|^2)\mathcal{B}(x,y),$$

where

$$\mathcal{B}(x,y) = (y_1 y_2 - x_1 x_2).$$

Here the matrix A has the form

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

and has double eigenvalues $\pm i$. If $x = x(t)$ and $y = y(t)$ is a solution of (HS), there results

$$\frac{d}{dt}(x_1 y_2 + y_1 x_2) = -4[\mathcal{B}(x,y)]^2 - (|x|^2 + |y|^2)^2,$$

and therefore (HS) has the trivial solution $x \equiv 0$, $y \equiv 0$ only.

(iv) Since b'' is non-singular, the arguments of Lemma 3.1 show that here the auxiliary parameter $\mu = 0$.

(v) According to Remark 1.3, the family of periodic solutions u_s has the property that

$$\frac{u_s}{s} \to \xi e^{i\omega t} + \xi^* e^{-i\omega t} \quad \text{as } s \to 0$$

where ξ is such that $A\xi = i\omega_0 \xi$. ∎

The Lyapunov Centre Theorem applies both to second-order gradient systems like (3.3) and to Hamiltonian Systems (HS). Let us state explicitly this kind of result.

We consider (HS) with $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. Set $z = (x,y) \in \mathbb{R}^{2n}$, $H(z) = H(x,y)$ and $\nabla H(z) = (H_x(z), H_y(z))$. If J denotes the symplectic matrix (i.e. $J : (x,y) \to (-y, x)$), then (HS) can be written in the more compact form

$$\frac{dz}{dt} = J\nabla H(z).$$

Since in (HS) the Hamiltonian H is a constant of the motion, namely

$H(z(t)) \equiv$const. for all solutions of (HS), it makes sense to look for periodic solutions of (HS) on the Hamiltonian surface $H(z) = h$.

We suppose that

(H0) $H(0) = 0$, $\nabla H(0) = 0$ and $H''(0) > 0$ (that is $H''(0)$ is positive-definite),

(H1) $JH''(0)$ has n pairs of purely imaginary simple eigenvalues

$$\pm \mathrm{i}\omega_k, \; k = 1, 2, \ldots, n,$$

such that ω_i/ω_j is not an integer for all $i \neq j$.

Theorem 3.4 *Suppose that* $H \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ *satisfies* (H0–1). *Then for all* $\varepsilon > 0$ *small enough* (HS) *has* n *(geometrically) distinct periodic orbits on the surface* $H(z) = \varepsilon$. *More precisely, the surface* $H(z) = \varepsilon$ *carries* n *distinct periodic orbits* z_k *whose periods tend to* $2\pi/\omega_k$, $k = 1, 2, \ldots, n$.

Proof. For all $k = 1, 2, \ldots, n$, we can apply Theorem 3.2 with $f = J\nabla H, A = JH''(0)$ and $b = H$. Indeed, (H0) implies, in particular, that $b''(0) = H''(0)$ is non-singular; and (H1) implies that (Ai–ii) hold true for all $k = 1, 2, \ldots, n$.

Then there exist n branches $z_{k,s}$, $k = 1, 2, \ldots, n$. of periodic solutions of (HS) with period $T_{k,s}$ converging to $2\pi/\omega_k$ as $s \to 0$; moreover,

$$\|z_{k,s}\|_{L^\infty} \to 0 \quad \text{as } s \to 0. \tag{3.7}$$

In addition, $z_{k,s}$ depends in a C^1 fashion on s and (see Remark 3.3(v))

$$\lim_{s \to 0} \frac{z_{k,s}}{s} = v_k := \xi_k e^{\mathrm{i}\omega t} + \xi_k^* e^{-\mathrm{i}\omega t}, \quad k = 1, 2, \ldots, n \tag{3.8}$$

where $A\xi_k = \mathrm{i}\omega_k \xi_k$.

Since H is a first integral of (HS), then $H(z_{k,s}(t))$ is independent of t. We set

$$h_k(s) = H(z_{k,s}(0)).$$

From (3.7) one immediately deduces that $h_k(s) \to H(0) = 0$.

Moreover, since $z_{k,s}$ is C^1 with respect to s and $H'(0) = 0$, it follows readily that h_k is twice differentiable at $s = 0$ and, using also (3.8), one finds

$$h_k''(0) = H''(0) w_k \cdot w_k$$

where $w_k = \xi_k + \xi_k^*$.

Since $H''(0) > 0$, it follows that for all $\varepsilon > 0$ small enough and any $k = 1, 2, \ldots, n$, the equation $h_k(s) = \varepsilon$ has a solution $s = s(k, \varepsilon)$ and

$$s(k, \varepsilon) \to 0 \text{ as } \varepsilon \to 0 \quad k = 1, 2, \ldots, n.$$

Correspondingly we find n solutions $z_{k,\varepsilon} = z_{k,s(k,\varepsilon)}$ $(k = 1,2,\ldots,n)$ of (HS) such that $H(z_{k,\varepsilon}) = \varepsilon$. Finally, from (3.8) we also deduce that, for ε small, the orbit of $z_{k,\varepsilon}$ is close to that of $s\tilde{v}_k$, that is to that of sv_k, up to higher-order terms; then the $z_{k,\varepsilon}$ $(k = 1,2,\ldots,n)$ correspond to *geometrically distinct orbits.* This completes the proof of the theorem.

Remarks 3.5

(i) As in Remark 3.3 (i), H could be defined in a neighbourhood of 0 in $\mathbf{R}^n$.

(ii) Theorem 3.4 has been extended by Weinstein [**W**] (see also [**Mo**]) who proved the following result. *Suppose H satisfies* (H0). *Then for all $\varepsilon > 0$ small enough* (HS) *has n distinct periodic orbits on the surface* $H(z) = \varepsilon$. In comparison with the result of Moser recalled in Remark 3.3 (iii), one has to point out that in the case of a general conservative system one can exibit examples where (S) has only one solution on each surface $b = \varepsilon$.

4 The restricted three-body problem

One of the most classical application of the Lyapunov Centre Theorem is to the existence of small oscillations near the equilibrium points of the planar restricted three-body problem. This problem deals with three-bodies P_1, P_2 (called *primaries*) and Q, with masses M_1, M_2 and M_3, respectively, under the action of the Newton Gravitational Law. To make the problem more feasible, one considers the *restricted problem,* which is concerned with the case when the mass of one particle is negligible with respect to the others. If, say, $M_3 = 0$ then P_1 and P_2 are not influenced by Q and they move according to the solutions of a two-body problem. Our aim is to study the motion of Q under the attraction of the two primaries.

Actually, we shall make some further simplifications. First of all, we suppose that the primaries move on circles, rather than more general elliptical orbits, with constant angular velocity γ. Moreover we will assume that the motion of Q occurs on the same plane as that of P_1, P_2. This problem is usually called the *restricted planar three-body problem.* Even with these simplifications, it is still quite interesting, because many problems arising in celestial mechanics fit in this frame.

Let us introduce a rotating coordinate system xOy (Fig. 7.6), such that (i) the origin O coincides with the barcentre of P_1 and P_2 and (ii) P_1 and P_2 are at rest on the x-axis. With a suitable choice of the units,

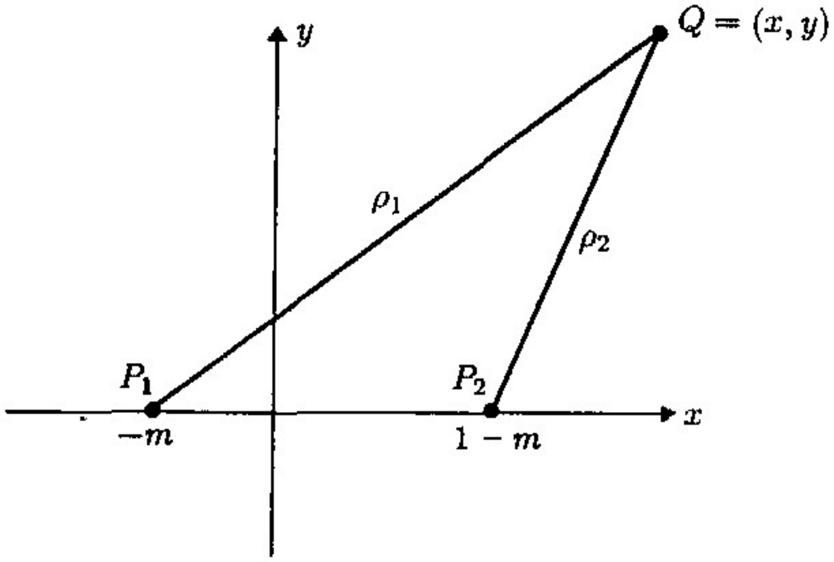

Figure 7.6

we can take $M_1 + M_2 = 1$, $\gamma = 1$ and g (the gravity constant) $= 1$. We also set $M_2 = m$ in such a way that $P_1 = (-m, 0)$ and $P_2 = (1-m, 0)$ and let (x, y) denote the coordinates of Q and

$$\rho_1 = \sqrt{[(x+m)^2 + y^2]},$$
$$\rho_2 = \sqrt{[(x+m-1)^2 + y^2]}$$

the distances from Q to P_1 and P_2, respectively.

The third body Q is subjected to combined action of the centrifugal and Coriolis forces and to those due to the Newtonian attraction, corresponding to the potential

$$U(x,y) = \frac{1-m}{\rho_1} + \frac{m}{\rho_2}.$$

In conclusion, we find tbe system

$$\left.\begin{aligned} x'' - 2y' - x &= U_x(x,y), \\ y'' + 2x' - y &= U_y(x,y), \end{aligned}\right\} \tag{4.1}$$

where, here and hereafter, primes $'$ denote $\mathrm{d}/\mathrm{d}t$.

Equilibrium points

The possible equilibria of (4.1) can be found by solving the system

$$\left.\begin{aligned} -x &= U_x(x,y), \\ -y &= U_y(x,y), \end{aligned}\right\}$$

namely the pair of equations

$$-x = -\frac{(x+m)(1-m)}{\rho_1^3} - \frac{m(x+m-1)}{\rho_2^3}, \tag{4.2}$$

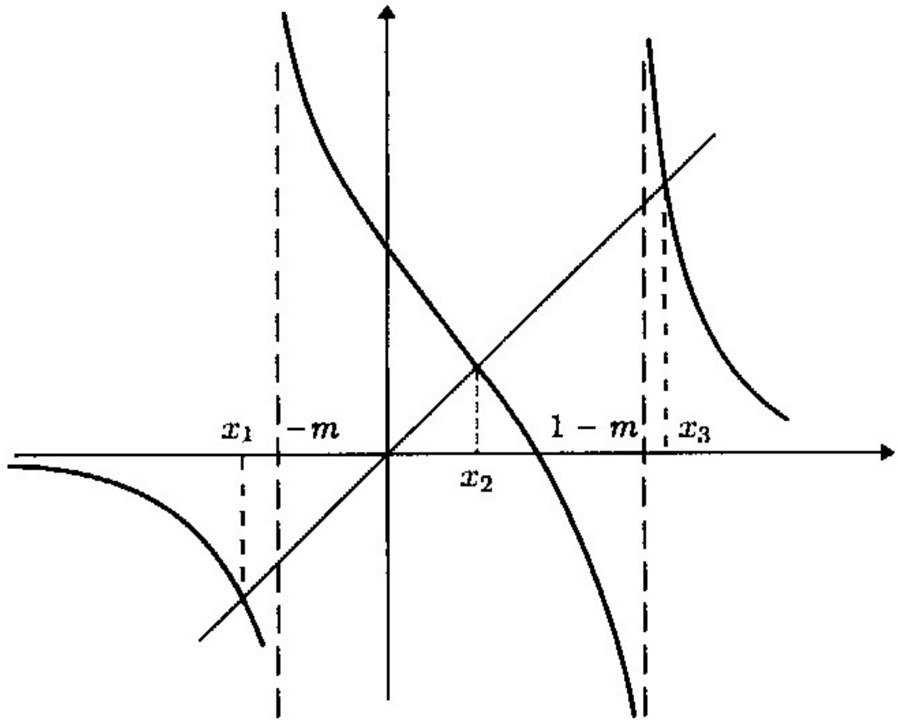

Figure 7.7

$$-y = -\frac{y(1-m)}{\rho_1^3} - \frac{my}{\rho_2^3}. \tag{4.3}$$

The lattor is satisfied for $y = 0$. Substituting $y = 0$ into (4.2) we find

$$x = \frac{(x+m)(1-m)}{|x+m|^3} + \frac{m(x+m-1)}{|x+m-1|^3}. \tag{4.4}$$

Equation (4.4) has three solutions, corresponding to the so called *Euler points* L_1, L_2, L_3. (Figure 7.7).

It is also convenient to introduce the potential

$$\Phi(x,y) = \frac{1}{2}(x^2+y^2) + U(x,y). \tag{4.5}$$

Equations (4.2) and (4.3) are nothing but $\Phi_x = 0$ and $\Phi_y = 0$, respectively. Hence the Euler points are the solutions of $\Phi_x(x,0) = 0$. One checks immediately that $\Phi_{xx}(L_i) > 0$, $i = 1,2,3$. As for Φ_{yy} one finds

$$\Phi_{yy}(x,0) = 1 - \frac{m}{|x-m+1|^3} - \frac{1-m}{|x-m|^3}.$$

Since in L_2 both $|x-m|$ and $|x-m+1|$ are < 1 we infer that $\Phi_{yy}(L_2) < 0$. In L_1 and L_3 one finds, with elementary calculations, that $x\Phi_{yy}(x,0)$ is < 0 in L_3 and > 0 in L_1. In both cases it follows that $\Phi_{yy} < 0$. In other words, letting for $1 \le j \le 3$

$$a_j = \Phi_{xx}(L_j),\ b_j = \Phi_{xy}(L_j),\ c_j = \Phi_{yy}(L_j),$$

we get

$$b_j = 0,\ a_j > 0 \text{ and } c_j < 0. \tag{4.6}$$

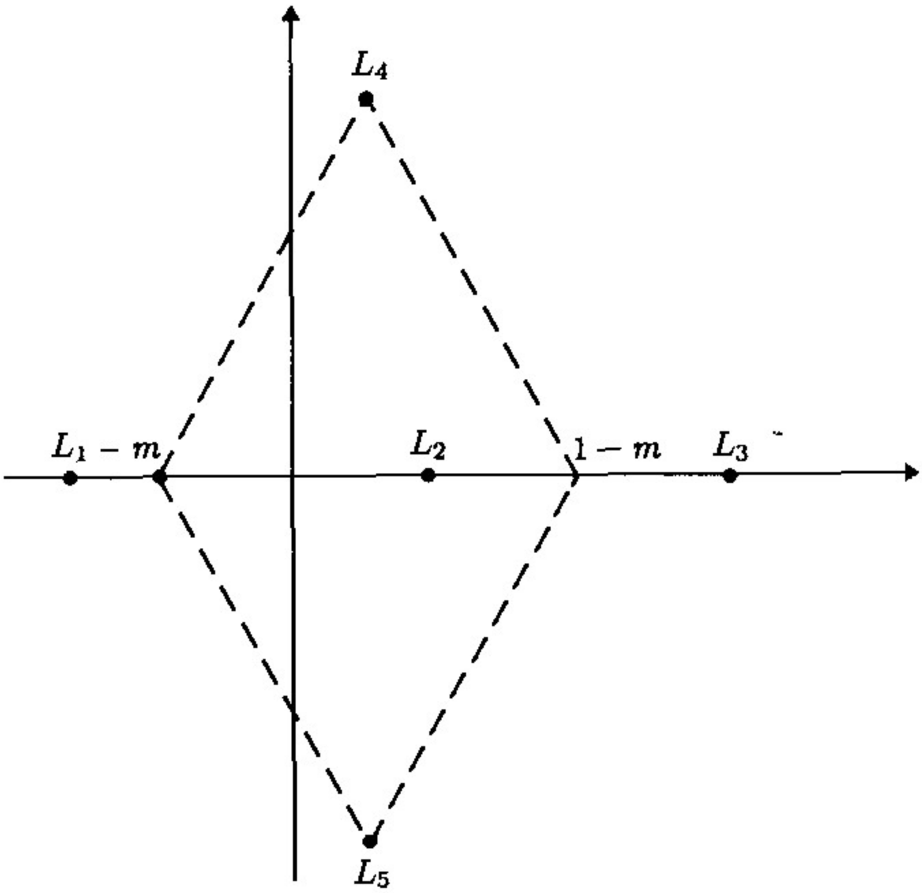

Figure 7.8

Let us come back to (4.2)–(4.3) and look for solutions with $y \neq 0$. Setting $h = 1/\rho_1^3$ and $k = 1/\rho_2^3$ we find

$$x = h(x+m)(1-m) + km(x+m-1), \tag{4.7}$$

$$1 = h(1-m) + km. \tag{4.7'}$$

Multiplying (4.7') by x and subtracting from (4.7) one has readily $h = k = 1$. Thus there are two more equilibria L_4 and L_5, the *Lagrangian points*, such that P_1, P_2 and L_4 (or L_5) are the vertices of an equilateral triangle (Figure 7.8).

Setting $a = a_{4,5} = \Phi_{xx}(L_{4,5}), b = b_{4,5} = \Phi_{x,y}(L_{4,5})$ and $c = c_{4,5} = \Phi_{yy}(L_{4,5})$, one finds readily

$$a = \frac{3}{4}, b = \frac{3\sqrt{2}}{4}(2m-1), c = \frac{9}{4}. \tag{4.8}$$

The configuration consisting of the two primaries and a Lagrange point is, for example, that of the system Sun-Jupiter-Trojans (the last are a group of asteroids).

Small oscillations

Let us refer to the system (4.1) which will be written in the form

$$\left.\begin{aligned} x'' - 2y' &= \Phi_x(x,y), \\ y'' + 2x' &= \Phi_y(x,y), \end{aligned}\right\} \tag{4.1$'$}$$

where Φ is given by (4.5). In order to apply the Lyapunov Centre Theorem, (4.1′) has to be transformed into a first-order system. If we set $p = x'$ and $q = y'$, (4.1′) becomes

$$\left.\begin{aligned} x' &= p, \\ y' &= q, \\ p' &= 2q + \Phi_x, \\ q' &= -2p + \Phi_y, \end{aligned}\right\} \tag{4.1$''$}$$

which is of the form $u' = f(u)$, where $u = (x, y, p, q) \in \mathbb{R}^4$ and f has components

$$f(x, y, p, q) = (p, q, 2q + \Phi_x, -2p + \Phi_y).$$

In terms of the new coordinates, the equilibria are given by

$$u_j = (x_j, y_j, 0, 0), \; 1 \le j \le 5, \; \text{where } (x_j, y_j) = L_j.$$

It is immediately verifiable the system (4.1″) has a first integral (the *Jacobi integral*) given by

$$J(x, y, p, q) = \frac{1}{2}(p^2 + q^2) - \Phi(x, y).$$

The Hessian $J''(u_j)$ is given by (we keep the notation introduced before)

$$J''(u_j) = \begin{bmatrix} -a_j & -b_j & 0 & 0 \\ -b_j & -c_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consequently

$$\det[J''(u_j)] = \begin{vmatrix} -a_j & -b_j \\ -b_j & -c_j \end{vmatrix} = a_j c_j - b_j^2.$$

Taking into account (4.6) and (4.9) we find

$$\det[J''(u_j)] < 0, \; \text{for } j = 1, 2, 3,$$

$$\det[J''(u_j)] = \frac{27}{4} m(1 - m) > 0, \; \text{for } j = 4, 5,$$

and in any case J'' is nonsingular at each equilibrium point.

It remains to evaluate the matrix $A_j = f'(x_j, y_j, 0, 0)$. We obtain

$$A_j = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_j & b_j & 0 & 2 \\ b_j & c_j & -2 & 0 \end{bmatrix}.$$

The eigenvalues λ of A_j satisfy the equation

$$\lambda^4 - (a_j + c_j - 4)\lambda^2 + D_j = 0, \tag{4.9}$$

with

$$D_j = \det \begin{pmatrix} -a_j & -b_j \\ -b_j & -c_j \end{pmatrix}.$$

On the Euler points L_j, $j = 1,2,3$, we have $b_j = 0$, and (4.9) becomes

$$\lambda^4 - (a_j + c_j - 4)\lambda^2 + a_j c_j = 0. \tag{4.9'}$$

Since in addition $a_j c_j < 0$, then (4.9′) has a unique pair of purely imaginary roots $\lambda = \pm i\omega_j$, $j = 1,2,3$, and the Lyapunov Centre Theorem applies without any further restriction yielding the following.

Theorem 4.1 *In a neighbourhood of the Euler points L_j, $j = 1,2,3$, the restricted planar three-body problem has a family of periodic solutions whose periods tend to $2\pi/\omega_j$.*

As for the Lagrangian point L_4 (the same holds for the symmetric one L_5) we have (see (4.8))

$$a = \frac{3}{4}, \quad b = \frac{3\sqrt{2}}{4}(2m-1), \quad c = \frac{9}{4}.$$

The (4.9) becomes

$$\lambda^4 + \lambda^2 + \frac{27}{4}m(1-m) = 0,$$

which possesses two pairs of imaginary roots $\pm i\omega'$, $\pm i\omega''$, with, say, $0 < \omega' < \omega''$, provided $\frac{27}{4}m(1-m) < \frac{1}{4}$, namely for all $0 < m < m_0$ (or $1 - m_0 < m < 1$) where $m_0 \approx 0.0385\ldots$ is the smallest roots of $27m(1-m) = 1$. Let us consider the range $0 < m < m_0$ (for example, in the case when the primaries are Sun and Jupiter, the mass ratio is $m \approx 1/1000$, a value which is widely in the range $(0, m_0 \approx 0.0385)$; the same for the Earth–Moon system where $m \approx 1/82 \approx 0.012$).

Taking $\omega_0 = \omega''$ we can apply the Lyapunov Centre Theorem directly, while when we consider the pulsation ω' we have to require the *non-resonance condition* $\omega''/\omega' \notin \mathbb{N}$, namely that $\omega'' \neq k\omega'$ for all $k \in \mathbb{N}$. This leads to excluding the solutions of

$$\left.\begin{aligned} &\omega''^2 = k^2\omega'^2, \\ &\omega''^2 + \omega'^2 = 1, \\ &\omega''^2\omega'^2 = \frac{27}{4}m(1-m). \end{aligned}\right\}$$

This system has solutions whenever m satisfies

$$\frac{27}{4}m(1-m) = \frac{k^2}{(1+k^2)^2}. \tag{4.10}$$

In conclusion, if m_k denotes the sequence of solutions of (4.10) such that $m_k \to 0$, we can still apply, for $m \neq m_k$, the Lyapunov Centre Theorem yielding the following.

Theorem 4.2 *Suppose that* $0 < m < m_0$*; then in a neighbourhood of the Lagrange points* $L_{4,5}$ *the restricted planar three-body problem has a family of periodic solutions whose periods tend to* $2\pi/\omega''$*; if further,* $m \neq m_k$*, then there exists a second family of periodic solutions whose periods tend to* $2\pi/\omega'$.

Note that these periodic solutions correspond to *bounded* trajectories in the inertial frame of reference. For other results on the restricted three-body problem, see for example [**SiM**].

Remark 4.3 The stability of the linearized system $u' = A_j u$, namely of

$$\left.\begin{aligned} x'' - 2y' &= a_j x + b_j y, \\ y'' + 2x' &= b_j x + c_j y, \end{aligned}\right\} \tag{4.11}$$

at the equilibria $L_j, j = 1, 2, 3, 4, 5$, can be easily discussed.

At the Euler points L_1, L_2, L_3 the matrix A_j has a real positive and a real negative eigenvalue. Thus $L_j (j = 1, 2, 3)$ are unstable equilibria for (4.11) and are said to be *linearly unstable*.

Unlike the preceding case, the matrices A_3, A_4 have, for $0 < m < m_0$, two pairs of purely imaginary eigenvalues. Therefore, at L_4, L_5, (4.11) has bounded orbits only and the Lagrangian points are said to be *linearly stable*.

The question of the (nonlinear) stability of L_4, L_5 is much more delicate. It has been shown there are three exceptional values $m_i (i = 1, 2, 3), 0 < m_1 < m_2 < m_3 < m_0$, such that for all $m \in (0, m_0), m \neq m_i$ $(i = 1, 2, 3)$, the Lagrangian points are stable in the sense of Lyapunov. For more details, see [**Mo1**].