Chapter 2

Theories and elementarity

1 Logical consequences

Questo paragrafo lo assumo noto da IdL
(a parte forse i tre esempi evidenziati)

A theory is a set $T \subseteq L$ of sentences. We write $M \vDash T$ if $M \vDash \varphi$ for every $\varphi \in T$. If $\varphi \in L$ is a sentence we write $T \vdash \varphi$ when

$$M \vDash T \quad \Rightarrow \quad M \vDash \varphi \qquad \text{for every } M.$$

In words, we say that φ is a logical consequence of T or that φ follows from T. If S is a theory $T \vdash S$ has a similar meaning. If $T \vdash S$ and $S \vdash T$ we say that T and S are logically equivalent. We may say that T axiomatizes S (or vice versa).

We say that a theory is consistent if it has a model. With the notation above, T is consistent if and only if $T \nvdash \bot$.

The closure of T under logical consequence is the set $\operatorname{ccl}(T)$ which is defined as follows:

$$\operatorname{ccl}(T) \quad = \quad \Big\{\varphi \in L \;:\; \text{sentence such that } T \vdash \varphi\Big\}$$

If T is a finite set, say $T = \{\varphi_1, \dots, \varphi_n\}$ we write $\operatorname{ccl}(\varphi_1, \dots, \varphi_n)$ for $\operatorname{ccl}(T)$. If $T = \operatorname{ccl}(T)$ we say that T is closed under logical consequences.

The theory of M is the set of sentences that hold in M and is denoted by $\operatorname{Th}(M)$. More generally, if $\mathcal{K}$ is a class of structures, $\operatorname{Th}(\mathcal{K})$ is the set of sentences that hold in every model in $\mathcal{K}$. That is

$$\operatorname{Th}(\mathcal{K}) \quad = \quad \bigcap_{M \in \mathcal{K}} \operatorname{Th}(M)$$

The class of all models of T is denoted by $\operatorname{Mod}(T)$. We say that $\mathcal{K}$ is axiomatizable if $\operatorname{Mod}(T) = \mathcal{K}$ for some theory T. If T is finite we say that $\mathcal{K}$ is finitely axiomatizable. To sum up

$$\begin{aligned}
\operatorname{Th}(M) &= \Big\{\varphi \;:\; M \vDash \varphi\Big\} \\
\operatorname{Th}(\mathcal{K}) &= \Big\{\varphi \;:\; M \vDash \varphi \text{ for all } M \in \mathcal{K}\Big\} \\
\operatorname{Mod}(T) &= \Big\{M \;:\; M \vDash T\Big\}
\end{aligned}$$

2.1 Example Let L be the language of multiplicative groups. Let T_g be the set containing the universal closure of following three formulas

1. $(x \cdot y) \cdot z \;=\; x \cdot (y \cdot z)$;
2. $x \cdot x^{-1} \;=\; x^{-1} \cdot x \;=\; 1$;
3. $x \cdot 1 \;=\; 1 \cdot x \;= x$.

Then T_g axiomatizes the theory of groups, i.e. $\operatorname{Th}(\mathcal{K})$ for $\mathcal{K}$ the class of all groups. Let φ be the universal closure of the following formula

$$z \cdot x = z \cdot y \to x = y.$$

As φ formalizes the cancellation property then $T_g \vdash \varphi$, that is, φ is a logical consequence of T_g. Now consider the sentence ψ which is the universal closure of

4. $x \cdot y = y \cdot x.$

So, commutative groups model ψ and non commutative groups model $\neg\psi$. Hence neither $T_g \vdash \psi$ nor $T_g \vdash \neg\psi$. We say that T_g does not decide ψ. □

Note that even when T is a very concrete set, $\mathrm{ccl}(T)$ may be more difficult to grasp. In the example above T_g contains three sentences but $\mathrm{ccl}(T_g)$ is an infinite set containing sentences that code theorems of group theory yet to be proved.

2.2 Remark The following properties say that ccl is a finitary closure operator.

1. $T \subseteq \mathrm{ccl}(T)$ (extensive)
2. $\mathrm{ccl}(T) = \mathrm{ccl}(\mathrm{ccl}(T))$ (idempotent)
3. $T \subseteq S \Rightarrow \mathrm{ccl}(T) \subseteq \mathrm{ccl}(S)$ (increasing)
4. $\mathrm{ccl}(T) = \bigcup \{\mathrm{ccl}(S) : S \text{ finite subset of } T\}.$ (finitary)

Properties 1-3 are easy to verify while 4 requires the compactness theorem. □

In the next example we list a few algebraic theories with straightforward axiomatization.

2.3 Example We write T_{ag} for the theory of abelian groups which contains the universal closure of following

a1. $(x + y) + z = y + (x + z);$

a2. $x + (-x) = 0;$

a3. $x + 0 = x;$

a4. $x + y = y + x.$

The theory T_{r} of (unitary) rings extends T_{ag} with

a5. $(x \cdot y) \cdot z = x \cdot (y \cdot z);$

a6. $1 \cdot x = x \cdot 1 = x;$

a7. $(x + y) \cdot z = x \cdot z + y \cdot z;$

a8. $z \cdot (x + y) = z \cdot x + z \cdot y.$

The theory of commutative rings T_{cg} contains also com of examples 2.1. The theory of ordered rings T_{or} extends T_{cr} with

o1. $x < z \to x + y < z + y;$

o2. $0 < x \wedge 0 < z \to 0 < x \cdot z.$ □

The axiomatization of the theory of vector spaces is less straightforward.

2.4 Example Fix a field F. The language L_F extends the language of additive groups with a unary function for every element of F. The theory of vector fields over F extends T_{ag} with the following axioms (for all $h, k, l \in F$)

m1. $h\,(x+y) \;=\; h\,x + h\,y$

m2. $l\,x \;=\; h\,x + k\,x,$ where $l \;=\; h +_F k$

m3. $l\,x \;=\; h\,(k\,x),$ where $l \;=\; h \cdot_F k$

m4. $0_F\,x \;=\; 0$

m5. $1_F\,x \;=\; x$

The symbols 0_F and 1_F denote the zero and the unit of F. The symbols $+_F$ and $\cdot_F$ denote the sum and the product in F. These are not part of L_F, they are symbols we use in the metalanguage. □

2.5 Example Recall from Example 1.3 that we represent a graph with a symmetric irreflexive relation. Therefore theory of graphs contains the following two axioms

1. $\neg r(x,x)$;
2. $r(x,y) \to r(y,x)$. □

Our last example is a trivial one.

2.6 Example Let L be the empty language The theory of infinite sets is axiomatized by the sentences $\exists^{\geq n} x\ (x = x)$ for all positive integer n. □

2.7 Exercise Prove that $\mathrm{ccl}(\varphi \vee \psi) = \mathrm{ccl}(\varphi) \cap \mathrm{ccl}(\psi)$. □

2.8 Exercise Prove that $T \cup \{\varphi\} \vdash \psi$ then $T \vdash \varphi \to \psi$. □

2.9 Exercise Prove that $\mathrm{Th}(\mathrm{Mod}(T)) = \mathrm{ccl}(T)$. □

2 Elementary equivalence Tutto

The following is a fundamental notion in model theory.

2.10 Definition *We say that M and N are elementarily equivalent if*

ee. $N \vDash \varphi \quad \Leftrightarrow \quad M \vDash \varphi,$ *for every sentence $\varphi \in L$.*

In this case we write $M \equiv N$. More generally, we write $M \equiv_A N$ and say that M and N are elementarily equivalent over A if the following hold

a. $A \subseteq M \cap N$

ee'. equivalence ee *above holds for every sentence $\varphi \in L(A)$.* □

The case when A is the whole domain of M is particularly important.

2.11 Definition *When $M \equiv_M N$ we write $M \preceq N$ and say that M is an elementary substructure of N.* □

In the definition above the use of the term *substructure* is appropriate by the following lemma.

2.12 Lemma *If M and N are such that $M \equiv_A N$ and A is the domain of a substructure of M then A is also the domain a substructure of N and the two substructures coincide.*

Proof Let f be a function symbol and ler r be a relation symbol. It suffices to prove that $f^M(a) = f^N(a)$ for every $a \in A^{n_f}$ and that $r^M \cap A^{n_r} = r^N \cap A^{n_r}$.

If $b \in A$ is such that $b = f^M a$ then $M \vDash fa = b$. So, from $M \equiv_A N$, we obtain $N \vDash fa = b$, hence $f^N a = b$. This proves $f^M(a) = f^N(a)$.

Now let $a \in A^{n_r}$ and suppose $a \in r^M$. Then $M \vDash ra$ and, by elementarity, $N \vDash ra$, hence $a \in r^N$. By symmetry $r^M \cap A^{n_r} = r^N \cap A^{n_r}$ follows. □

It is not easy to prove that two structures are elementary equivalent. A direct verification is unfeasible even for the most simple structures. It will take a few chapters before we are able to discuss concrete examples.

We generalize the definition of Th(M) to include parameters

$$\text{Th}(M/A) = \Big\{\varphi \ : \ \text{sentence in } L(A) \text{ such that } M \vDash \varphi\Big\}.$$

The following proposition is immediate

2.13 Proposition *For every pair of structures M and N and every $A \subseteq M \cap N$ the following are equivalent*

a. $M \equiv_A N$;

b. $\text{Th}(M/A) = \text{Th}(N/A)$;

c. $M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(a)$ *for every $\varphi(x) \in L$ and every $a \in A^{|x|}$.*

d. $\varphi(M) \cap A^{|x|} = \varphi(N) \cap A^{|x|}$ *for every $\varphi(x) \in L$.* □

If we restate a and c of the proposition above when $A = M$ we obtain that the following are equivalent

a'. $M \preceq N$;

d'. $\varphi(M) = \varphi(N) \cap M^{|x|}$ for every $\varphi(x) \in L$.

Note that c' extends to all definable sets what Definition 1.7 requires for a few basic definable sets.

2.14 Example Let G be a group which we consider as a structure in the multiplicative language of groups. We show that if G is simple and $H \preceq G$ then also H is simple. Recall that G is simple if all its normal subgroups are trivial, equivalently, if for every $a \in G \setminus \{1\}$ the set $\{gag^{-1} : g \in G\}$ generates the whole group G.

Assume H is not simple. Then there are $a, b \in H$ such that b is not the product of elements of $\{hah^{-1} : h \in H\}$. Then for every n

$$H \vDash \neg\exists x_1, \dots, x_n \ \left(b = x_1 a x_1^{-1} \cdots x_n a x_n^{-1}\right)$$

By elementarity the same hold in G. Hence G is not simple. □

2.15 Exercise Let $A \subseteq M \cap N$. Prove that $M \equiv_A N$ if and only if $M \equiv_B N$ for every finite $B \subseteq A$. Facile ma importante □

2.16 Exercise Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if

$\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$. □

2.17 Exercise Let $M \preceq N$ and let $\varphi(x,z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M. □

2.18 Exercise Consider $\mathbb{Z}^n$ as a structure in the additive language of groups with the natural interpretation. Prove that $\mathbb{Z}^n \not\equiv \mathbb{Z}^m$ for every positive integers $n \neq m$. Hint: in $\mathbb{Z}^n$ there are at most 2^n elements that are not congruent modulo 2. □

3 Embeddings and isomorphisms

La definizione di omomorfismo la rimandiamo

Here we prove that isomorphic structures are elementarily equivalent and a few related results.

2.19 Definition *An embedding of M into N is an injective total map $h : M \hookrightarrow N$ such that*

1. $a \in r^M \quad \Leftrightarrow \quad ha \in r^N$ *for every $r \in L_{\text{rel}}$ and $a \in M^{n_r}$;*
2. $h\, f^M(a) \quad = \quad f^N(h\, a)$ *for every $f \in L_{\text{fun}}$ and $a \in M^{n_f}$.*

Note that when $c \in L_{\text{fun}}$ is a constant 2 reads $h\, c^M = c^N$. Therefore that $M \subseteq N$ if and only if $\text{id}_M : M \to N$ is an embedding.

An surjective embedding is an isomorphism or, when domain and codomain coincide, an automorphism. □

Condition 1 above and the assumption that h is injective can be summarized in the following

1'. $M \vDash r(a) \quad \Leftrightarrow \quad N \vDash r(ha)$ for every $r \in L_{\text{rel}} \cup \{=\}$ and every $a \in M^{n_r}$.

Note also that, by straightforward induction on syntax, from 2 we obtain

2' $h\, t^M(a) \quad = \quad t^N(h\, a)$ for every term $t(x)$ and every $a \in M^{|x|}$.

Combining these two properties and a straightforward induction on the syntax give

3. $M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha)$ for every $\varphi(x) \in L_{\text{qf}}$ and every $a \in M^{|x|}$.

Recall that we write L_{qf} for the set of quantifier-free formulas. It is worth noting that when $M \subseteq N$ and $h = \text{id}_M$ then 3 becomes

3' $M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(a)$ for every $\varphi(x) \in L_{\text{qf}}$ and for every $a \in M^{|x|}$.

In words this is summarized by saying that the truth of quantifier-free formulas is preserved under sub- and superstructure.

Finally we prove that first order truth is preserved under isomorphism. We say that a map $h : M \to N$ fixes $A \subseteq M$ (pointwise) if $\text{id}_A \subseteq h$. An isomorphism that fixes A is also called an A-isomorphism.

2.20 Theorem *If $h : M \to N$ is an isomorphism then for every $\varphi(x) \in L$*

\# $\quad M \vDash \varphi(a) \quad \Leftrightarrow \quad N \vDash \varphi(ha)$ *for every $a \in M^{|x|}$*

In particular, if h is an A-isomorphism then $M \equiv_A N$.

Proof We proceed by induction of the syntax of $\varphi(x)$. When $\varphi(x)$ is atomic # holds by 3 above. Induction for the Boolean connectives is straightforward so we only need to consider the existential quantifier. Assume as induction hypothesis that

$$M \vDash \varphi(a,b) \quad \Leftrightarrow \quad N \vDash \varphi(ha,hb) \qquad \text{for every tupla } a \in M^{|x|} \text{ and } b \in M.$$

We prove that # holds for the formula $\exists y\, \varphi(x,y)$.

$$\begin{aligned} M \vDash \exists y\, \varphi(a,y) \quad &\Leftrightarrow \quad M \vDash \varphi(a,b) \quad \text{for some } b \in M \\ &\Leftrightarrow \quad N \vDash \varphi(ha,hb) \text{ for some } b \in M \quad \text{(by induction hypothesis)} \\ &\Leftrightarrow \quad N \vDash \varphi(ha,c) \quad \text{for some } c \in N \qquad (\Leftarrow \text{ by surjectivity}) \\ &\Leftrightarrow \quad N \vDash \exists y\, \varphi(ha,y). \end{aligned}$$

□

2.21 Corollary *If $h : M \to N$ is an isomorphism then $h\big[\varphi(M)\big] = \varphi(N)$ for every $\varphi(x) \in L$.* □

We can now give a few very simple examples of elementarily equivalent structures.

2.22 Example Let L be the language of strict orders. Consider intervals of $\mathbb{R}$ (or in $\mathbb{Q}$) as structures in the natural way. The intervals $[0,1]$ and $[0,2]$ are isomorphic, hence $[0,1] \equiv [0,2]$ follows from Theorem 2.20. Clearly, $[0,1]$ is a substructure of $[0,2]$. However $[0,1] \not\preceq [0,2]$, in fact the formula $\forall x\, (x{\le}1)$ holds in $[0,1]$ but is false in $[0,2]$. This shows that $M \subseteq N$ and $M \equiv N$ does not imply $M \preceq M$.

Now we prove that $(0,1) \preceq (0,2)$. By Exercise 2.15 above, it suffices to verify that $(0,1) \equiv_B (0,2)$ for every finite $B \subseteq (0,1)$. This follows again by Theorem 2.20 as $(0,1)$ and $(0,2)$ are B-isomorphic for every finite $B \subseteq (0,1)$. □

For the sake of completeness we also give the definition of homomorphism.

2.23 Definition *A homomorphism is a total map $h : M \hookrightarrow N$ such that*

1. $a \in r^M \quad \Rightarrow \quad ha \in r^N \qquad$ *for every $r \in L_{\text{rel}}$ and $a \in M^{n_r}$;*

2. $h\, f^M(a) \quad = \quad f^N(h\,a) \qquad$ *for every $f \in L_{\text{fun}}$ and $a \in M^{n_f}$.*

Note that only one implication is required in 1. □

2.24 Exercise Prove that if $h : N \to N$ is an automorphism and $M \preceq N$ then $h[M] \preceq N$. □

2.25 Exercise Let L be the empty language. Let $A, D \subseteq M$. Prove that the following are equivalent

1. D is definable over A;
2. either D is finite and $D \subseteq A$, or $\neg D$ is finite and $\neg D \subseteq A$.

Hint: as structures are plain sets, every bijection $f : M \to M$ is an automorphism. □

2.26 Exercise Prove that if $\varphi(x)$ is an existential formula and $h : M \hookrightarrow N$ is an embedding then

$$M \vDash \varphi(a) \quad \Rightarrow \quad N \vDash \varphi(ha) \qquad \text{for every } a \in M^{|x|}.$$

Recall that existential formulas as those of the form $\exists y\, \psi(x,y)$ for $\psi(x,y) \in L_{\text{qf}}$. Note that Theorem 10.7 proves that the property above characterizes existential formulas. □

2.27 Exercise Let M be the model with domain $\mathbb{Z}$ in the language that contains only the symbol $+$ which is interpreted in the usual way. Prove that there is no existential formula $\varphi(x)$ such that $\varphi(M)$ is the set of odd integers. Hint: use Exercise 2.26. □

2.28 Exercise Let N be the multiplicative group of $\mathbb{Q}$. Let M be the subgroup of those rational numbers that are of the form n/m for some odd integers m and n. Prove that $M \preceq N$. Hint: use the fundamental theorem of arithmetic and reason as in Example 2.22. □

4 Quotient structures Rimandato

The content of this section is mainly technical and only required later in the course. Its reading may be postponed.

If E is an equivalence relation on N we write $[c]_E$ for the equivalence class of $c \in N$. We use the same symbol for the equivalence relation on N^n defined as follow: if $a = a_1, \ldots, a_n$ and $b = b_1, \ldots, b_n$ are n-tuples of elements of N then $a \mathrel{E} b$ means that $a_i \mathrel{E} b_i$ holds for all i. It is easy to see that $b_1, \ldots, b_n \in [a, ..., a_n]_E$ if and only if $b_i \in [a_i]_E$ for all i. Therefore we use the notation $[a]_E$ for both the equivalence class of $a \in N^n$ and the tuple of equivalence classes $[a_1]_E, \ldots, [a_n]_E$.

2.29 Definition *We say that the equivalence relation E on a structure N is a congruence if for every $f \in L_{\text{fun}}$*

c1. $$a \mathrel{E} b \quad \Rightarrow \quad f^N a \mathrel{E} f^N b;$$

When E is a congruence on N we write N/E for the a structure that has as domain the set of E-equivalence classes in N and the following interpretation of $f \in L_{\text{fun}}$ and $r \in L_{\text{rel}}$:

c2. $$f^{N/E}[a]_E \quad = \quad \left[f^N a\right]_E;$$

c3. $$[a]_E \ \in \ r^{N/E} \quad \Leftrightarrow \quad [a]_E \cap r^N \neq \varnothing.$$

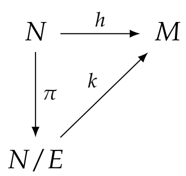
We call N/E the quotient structure. □

By c1 the quotient structure is well defined. The reader will recognize it as a familiar notion by the following proposition (which is not required in the following and requires the notion of homomorphism, see Definition 2.23. Recall that the kernel of a total map $h : N \to M$ is the equivalence relation E such that

$$a \mathrel{E} b \quad \Leftrightarrow \quad ha = hb$$

for every $a, b \in N$.

2.30 Proposition *Let $h : N \to M$ be a surjective homomorphism and let E be the kernel of h. Then there is an isomorphism k that makes the following diagram commute*



where $\pi : a \mapsto [a]_E$ is the projection map. □

⚠ Quotients clutter the notation with brackets. To avoid the mess, we prefer to reason in N and tweak the satisfaction relation. Warning: this is not standard (though it is what we all do all the time, informally).

Recall that in model theory, equality is not treated as a all other predicates. In fact, the interpretation of equality is fixed to always be the identity relation. In a few contexts is convinient to allow any congruence to interprete equality. This allows to work in N while thinking of N/E.

We define $N/E \stackrel{*}{\models}$ to be $N \models$ but with equality interpreted with E. The proposition below shows that this is the same thing as the regular truth in the quotient structure, $N/E \models$.

2.31 Definition *For t_2, t_2 closed terms of $L(N)$ define*

1* $\quad N/E \stackrel{*}{\models} t_1 = t_2 \quad \Leftrightarrow \quad t_1^N \, E \, t_2^N$

For t a tuple of closed terms of $L(N)$ and $r \in L_{\mathrm{rel}}$ a relation symbol

2* $\quad N/E \stackrel{*}{\models} r\,t \quad \Leftrightarrow \quad t^N \, E \, a \quad$ *for some $a \in r^N$*

Finally the definition is extended to all sentences $\varphi \in L(N)$ by induction in the usual way

3* $\quad N/E \stackrel{*}{\models} \neg\varphi \quad \Leftrightarrow \quad$ *not* $\; N/E \stackrel{*}{\models} \varphi$

4* $\quad N/E \stackrel{*}{\models} \varphi \wedge \psi \quad \Leftrightarrow \quad N/E \stackrel{*}{\models} \varphi$ *and* $\; N/E \stackrel{*}{\models} \psi$

5* $\quad N/E \stackrel{*}{\models} \exists x\, \varphi(x) \quad \Leftrightarrow \quad N/E \stackrel{*}{\models} \varphi(a) \quad$ *for some $a \in N$.* □

Now, by induction on the syntax of formulas one can prove $\stackrel{*}{\models}$ does what required. In particular, $N/E \stackrel{*}{\models} \varphi(a) \leftrightarrow \varphi(b)$ for every $a \, E \, b$.

2.32 Proposition *Let E be a congruence relation of N. Then the following are equivalent for every $\varphi(x) \in L$*

1. $\quad N/E \stackrel{*}{\models} \varphi(a)$;
2. $\quad N/E \models \varphi([a]_E)$. □

5 Completeness

Assumo sia tutto noto da IdL

A theory T is maximally consistent if it is consistent and there is no consistent theory S such that $T \subset S$. Equivalently, T contains every sentence φ consistent with T, that is, such that $T \cup \{\varphi\}$ is consistent. Clearly a maximally consistent theory is closed under logical consequences.

A theory T is complete if ccl$T$ is maximally consistent. Concrete examples will be given in the next chapters as it is not easy to prove that a theory is complete.

2.33 Proposition *The following are equivalent*

a. *T is maximally consistent;*

b. *$T = \mathrm{Th}(M)$ for some structure M;*

c. *T is consistent and $\varphi \in T$ or $\neg\varphi \in T$ for every sentence φ.*

Proof To prove a⇒b, assume that T is consistent. Then there is $M \vDash T$. Therefore $T \subseteq \text{Th}(M)$. As T is maximally consistent $T = \text{Th}(M)$. Implication b⇒c is immediate. As for c⇒a note that if $T \cup \{\varphi\}$ is consistent then $\neg\varphi \notin T$ therefore $\varphi \in T$ follows from c. □

The proof of the proposition below is is left as an exercise for the reader.

2.34 Proposition *The following are equivalent*

a. *T is complete;*

b. *there is a unique maximally consistent theory S such that $T \subseteq S$;*

c. *T is consistent and $T \vdash \text{Th}(M)$ for every $M \vDash T$;*

d. *T is consistent and $T \vdash \varphi$ o $T \vdash \neg\varphi$ for every sentence φ;*

e. *T is consistent and $M \equiv N$ for every pair of models of T.* □

2.35 Exercise Prove that the following are equivalent

a. T is complete;

b. for every sentence φ, o $T \vdash \varphi$ o $T \vdash \neg\varphi$ but not both.

By contrast prove that the following are *not* equivalent

a. T is maximally consistent;

b. for every sentence φ, o $\varphi \in T$ o $\neg\varphi \in T$ but not both.

Hint: consider the theory containing all sentences where the symbol $\neg$ occurs an even number of times. This theory is not consistent as it contains $\bot$. □

2.36 Exercise Prove that if T has exactly 2 maximally consistent extension T_1 and T_2 then there is a sentence φ such that $T, \varphi \vdash T_1$ and $T, \neg\varphi \vdash T_2$. State and prove the generalization to finitely many maximally consistent extensions. □

6 The Tarski-Vaught test Fatto a IdL, ma da ripassare.

There is no natural notion of *smallest* elementary substructure containing a set of parameters A. The downward Löwenheim-Skolem, which we prove in the next section, is the best result that holds in full generality. Given an arbitrary $A \subseteq N$ we shall construct a model $M \preceq N$ containing A that is small in the sense of cardinality. The construction selects one by one the elements of M that are required to realise the condition $M \preceq N$. Unfortunately, Definition 2.11 supposes full knowledge of the truth in M and it may not be applied during the construction. The following lemma comes to our rescue with a property equivalent to $M \preceq N$ that only mention the truth in N.

2.37 Lemma(Tarski-Vaught test) *For every $A \subseteq N$ the following are equivalent*

1. *A is the domain of a structure $M \preceq N$;*

2. *for every formula $\varphi(x) \in L(A)$, with $|x| = 1$,*
   $N \vDash \exists x\, \varphi(x) \quad \Rightarrow \quad N \vDash \varphi(b)$ *for some $b \in A$.*

Proof 1⇒2

$$\begin{aligned} N \vDash \exists x\, \varphi(x) &\Rightarrow M \vDash \exists x\, \varphi(x) \\ &\Rightarrow M \vDash \varphi(b) \qquad \text{for some } b \in M \\ &\Rightarrow N \vDash \varphi(b) \qquad \text{for some } b \in M. \end{aligned}$$

2⇒1 Firstly, note that A is the domain of a substructure of N, that is, $f^N a \in A$ for every $f \in L_{\mathrm{fun}}$ and every $a \in A^{n_f}$. In fact, this follows from 2 with $fa = x$ for $\varphi(x)$.

Write M for the substructure of N with domain A. By induction on the syntax we prove that for every $\xi(x) \in L$

$$M \vDash \xi(a) \Leftrightarrow N \vDash \xi(a) \qquad \text{for every } a \in M^{|x|}.$$

If $\xi(x)$ is atomic the claim follows from $M \subseteq N$ and the remarks underneath Definition 2.19. The case of Boolean connectives is straightforward, so only the existential quantifier requires a proof. So, let $\xi(x)$ be the formula $\exists y\, \psi(x, y)$ and assume the induction hypothesis holds for $\psi(x, y)$

$$\begin{aligned} M \vDash \exists y\, \psi(a, y) &\Leftrightarrow M \vDash \psi(a, b) \qquad \text{for some } b \in M \\ &\Leftrightarrow N \vDash \psi(a, b) \qquad \text{for some } b \in M \\ &\Leftrightarrow N \vDash \exists y\, \psi(a, y). \end{aligned}$$

The second equivalence holds by induction hypothesis, in the last equivalence we use 2 for the implication ⇐. □

2.38 Exercise Prove that, in the language of strict orders, $\mathbb{R} \smallsetminus \{0\} \preceq \mathbb{R}$ and $\mathbb{R} \smallsetminus \{0\} \not\preceq \mathbb{R}$. □

7 Downward Löwenheim-Skolem

Fatto a IdL, ma da ripassare.

The main theorem of this section was proved by Löwenheim at the beginning of the last century. Skolem gave a simpler proof immediately afterwards. At the time, the result was perceived as paradoxical.

A few years earlier, Zermelo and Fraenkel provided a formalization of set theory in a first order language. The downward Löwenheim-Skolem theorem implies the existence of an infinite countable model M of set theory: this is the so-called Skolem paradox. The existence of M seems paradoxical because, in particular, a sentence that formalises the axiom of power set holds in M. Therefore M contains an element b which, in M, is the set of subsets of the natural numbers. But the set of elements of b is a subset of M, and therefore it is countable.

In fact, this is not a contradiction, because the expression *all subsets of the natural numbers* does not have the same meaning in M as it has in the real world. The notion of cardinality, too, acquires a different meaning. In the language of set theory, there is a first order sentence that formalises the fact that b is uncountable: the sentence says that there is no bijection between b and the natural numbers. Therefore the bijection between the elements of b and the natural numbers (which exists in the real world) does not belong to M. The notion of equinumerosity has a different meaning in M and in the real world, but those who live in M cannot realise this.

2.39 Downward Löwenheim-Skolem Theorem *Let N be an infinite structure and fix some* *set $A \subseteq N$. Then there is a structure M of cardinality $\leq |L(A)|$ such that $A \subseteq M \preceq N$.*

Proof Set $\lambda = |L(A)|$. Below we construct a chain $\langle A_i : i < \omega \rangle$ of subsets of N. The chain begins at $A_0 = A$. Finally we set $M = \bigcup_{i<\omega} A_i$. All A_i will have cardinality $\leq \lambda$ so $|M| \leq \lambda$ follows.

Now we construct A_{i+1} given A_i. Assume as induction hypothesis that $|A_i| \leq \lambda$. Then $|L(A_i)| \leq \lambda$. For some fixed variable x let $\langle \varphi_k(x) \ : \ k < \lambda \rangle$ be an enumeration of the formulas in $L(A_i)$ that are consistent in N. For every k pick $a_k \in N$ such that $N \vDash \varphi_k(a_k)$. Define $A_{i+1} = A_i \cup \{a_k \ : \ k < \lambda\}$. Then $|A_{i+1}| \leq \lambda$ is clear.

We use the Tarski-Vaught test to prove $M \preceq N$. Suppose $\varphi(x) \in L(M)$ is consistent in N. As finitely many parameters occur in formulas, $\varphi(x) \in L(A_i)$ for some i. Then $\varphi(x)$ is among the formulas we enumerated at stage i and $A_{i+1} \subseteq M$ contains a solution of $\varphi(x)$. □

We will need to adapt the construction above to meet more requirements on the model M. To better control the elements that end up in M it is convenient to add one element at the time (above we add λ elements at each stage). We need to enumerate formulas with care if we want to complete the construction by stage λ.

2.40 Second proof of the downward Löwenheim-Skolem Theorem From set theory we know that there is a bijection $\pi : \lambda^2 \to \lambda$ such that $j, k \leq \pi(j,k)$ for all $j, k < \lambda$. Suppose we have defined the sets A_j for every $j \leq i$ and let $\langle \varphi^j_k(x) \ : \ k < \lambda \rangle$ be an enumeration of the consistent formulas of $L(A_j)$. Let $j, k \leq i$ be such that $\pi(j,k) = i$. Let b be a solution of the formula $\varphi^j_k(x)$ and define $A_{i+1} = A_i \cup \{b\}$.

We use Tarski-Vaught test to prove $M \preceq N$. Let $\varphi(x) \in L(M)$ be consistent in N. Then $\varphi(x) \in L(A_j)$ for some j. Then $\varphi(x) = \varphi^j_k$ for some k. Hence a witness of $\varphi(x)$ is enumerated in M at stage $\pi(j,k) + 1$. □

2.41 Exercise Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: adapt the construction used to prove the downward Löwenheim-Skolem Theorem. □

8 Elementary chains Fatto a IdL? Comunque per il momento è rimandata.

An elementary chain is a chain $\langle M_i \ : \ i < \lambda \rangle$ of structures such that $M_i \preceq M_j$ for every $i < j < \lambda$. The union (or limit) of the chain is the structure with as domain the set $\bigcup_{i<\lambda} M_i$ and as relations and functions the union of the relations and functions of M_i. It is plain that all structures in the chain are substructures of the limit.

2.42 Lemma *Let $\langle M_i : i \in \lambda \rangle$ be an elementary chain of structures. Let N be the union of the chain. Then $M_i \preceq N$ for every i.*

Proof By induction on the syntax of $\varphi(x) \in L$ we prove

$$M_i \vDash \varphi(a) \ \Leftrightarrow \ N \vDash \varphi(a) \qquad \text{for every } i < \lambda \text{ and every } a \in M_i^{|x|}$$

As remarked in 3' of Section 3, the claim holds for quantifier-free formulas. Induction for Boolean connectives is straightforward so we only need to consider the existential quantifier

$$\begin{aligned} M_i \vDash \exists y\, \varphi(a,y) \;&\Rightarrow\; M_i \vDash \varphi(a,b) \qquad \text{for some } b \in M_i. \\ &\Rightarrow\; N \vDash \varphi(a,b) \qquad \text{for some } b \in M_i \subseteq N \end{aligned}$$

where the second implication follows from the induction hypothesis. Vice versa

$$N \vDash \exists y\, \varphi(a,y) \;\Rightarrow\; N \vDash \varphi(a,b) \qquad \text{for some } b \in N$$

Without loss of generality we can assume that $b \in M_j$ for some $j \geq i$ and obtain

$$\Rightarrow\; M_j \vDash \varphi(a,b) \qquad \text{for some } b \in M_j$$

Now apply the induction hypothesis to $\varphi(x,y)$ and M_j

$$\begin{aligned} &\Rightarrow\; M_j \vDash \exists y\, \varphi(a,y) \\ &\Rightarrow\; M_i \vDash \exists y\, \varphi(a,y) \end{aligned}$$

where the last implication holds because $M_i \preceq M_j$. □

2.43 Exercise Let $\langle M_i : i \in \lambda \rangle$ be an chain of elementary substructures of N. Let M be the union of the chain. Prove that $M \preceq N$ and note that Lemma 2.42 is not required. □

2.44 Exercise Give an alternative proof of Exercise 2.41 using the downward Löwenheim-Skolem Theorem (instead of its proof). Hint: construct two countable chains of countable models such that $K_i \cap M \subseteq M_i \preceq N$ and $A \cup M_i \subseteq K_{i+1} \preceq N$. The required model is $K = \bigcup_{i \in \omega} K_i$. In fact it is easy to check that $K \cap M = \bigcup_{i \in \omega} M_i$. □