3 An axample: nonstandard analysis

This section concerns an example that is useful to look at in some detail to get some familiarity with the notion of elementary substructure. The example is culturally interesting, because it formalises rigorously the notions of infinity and infinitesimal, which were used in Newton and Leibniz's time to develop real analysis. These notions were not well defined - in fact, they were inconsistent.

It was only in the mid 19th century that mathematicians of Weiertsrass' generation developed the notion of limit, thus providing rigorous grounds for the development of analysis.

Nonstandard analysis was developed by Abraham Robinson in the 1950s. Robinson found a way to formalise the ideas of infinity and infinitesimals through the concept of elementary extension.

The notation that follows will be used throughout this section.

The language we use contains

1. X, a relation symbol of arity n, for every $n \in \omega$ and every $X \subseteq \mathbb{R}^n$;
2. f, a function symbol of arity n, for every $n \in \omega$ and every $f : \mathbb{R}^n \to \mathbb{R}$.

The standard model of real analysis is $\mathbb{R}$ with the natural interpretation of the symbols in our language, so we use the same symbol for an element of the language and its interpretation in $\mathbb{R}$. This is not an abuse of notation: in this case, elements and interpretations coincide.

Suppose $\mathbb{R}$ has a proper elementary extension ${}^*\mathbb{R}$. The existence of such an extension will be proved later. The interpretations of the symbols f and X in ${}^*\mathbb{R}$ will be denoted by ${}^*f$ and ${}^*X$, respectively. The elements of ${}^*\mathbb{R}$ are called hyperreals, the elements of $\mathbb{R}$ are called standard (hyper)reals, those in ${}^*\mathbb{R} \smallsetminus \mathbb{R}$ are called nonstandard (hyper)reals.

It is easy to verify that ${}^*\mathbb{R}$ is an ordered field. This is because the operations sum and product are in the language, and so is the order relation. The property of being an ordered field can be expressed via a set of sentences that are true in $\mathbb{R}$, and hence also in ${}^*\mathbb{R}$.

A hyperreal c is said to be infinitesimal if $|c| < \varepsilon$ for every positive standard ε. A hyperreal c is infinite if $k < |c|$ for every standard k; the hyperreal c if finite otherwise. Hence if c is infinite, then c^{-1} is infinitesimal. Clearly, all standard reals are finite, and 0 is the only infinitesimal standard real.

2.19 Lemma *There are infinite nonstandard hyperreals and nonzero infinitesimals.*

Proof Let $c \in {}^*\mathbb{R} \smallsetminus \mathbb{R}$ and suppose c is not infinite — otherwise, c and c^{-1} prove the lemma. Then the set $\{a \in \mathbb{R} : c < a\}$ is a nonempty set of reals that is bounded below. Let $b \in \mathbb{R}$ the least upper bound. We show that $b - c$ is a nonzero infinitesimal. We have that $b - c \neq 0$ because $c \in {}^*\mathbb{R} \smallsetminus \mathbb{R}$ and $b \in \mathbb{R}$. Assume for a contradiction that $\varepsilon < |b - c|$ for some standard positive ε. Then $c < b - \varepsilon$, or $b + \varepsilon < c$, depending on whether $c < b$ or $b < c$. Both cases contradict that b is an infimum. $\square$

As it happens, the proof above shows the following.

2.20 Corollary (of the proof) *For every finite hyperreal c there is a standard b such that $|c - b|$ is infinitesimal.* □

The existence of infinite non standard hyperreals shows that ${}^*\mathbb{R}$ is not archimedean: standard integers are not cofinal in ${}^*\mathbb{R}$. However, by elementarity, the nonstandard integers ${}^*\mathbb{N}$ are cofinal in ${}^*\mathbb{R}$: from the perspective of an inhabitant of ${}^*\mathbb{R}$, the latter is a normal archimedean field.

The proof of the following lemma is left to the reader.

2.21 Lemma *Infinitesimals are closed under sum, product and multiplication by standard reals.*

Dedekind completeness is another fundamental property of $\mathbb{R}$ that does not hold in ${}^*\mathbb{R}$. An ordered set is Dedekind complete if every subset that is bounded above has a least upper bound. Then ${}^*\mathbb{R}$ is not Dedekind complete: the set of infinitesimals is bounded (both above and below), but, by Lemma 2.21, it does not have a least upper bound. It follows that Dedekind completeness is *not* a first-order property.

However, it sometimes happens that properties that hold of all sets in the standard model hold in ${}^*\mathbb{R}$ for definable sets only.

2.22 Exercise Every definable (possibly with parameters) subset of ${}^*\mathbb{R}$ that is bounded above has a least upper bound. □

We define the following equivalence relation on ${}^*\mathbb{R}$: we write $a \approx b$ if $|a - b|$ is infinitesimal. The fact that this is an equivalence relation follows from Lemma 2.21. The equivalence class of c is called monad.

2.23 Lemma *If c is a finite hyperreal, there is a unique real in the monad of c.*

Proof Existence follows from Corillary 2.20. For uniqueness, observe that if $b_1 \approx b_2$ are both standard, then $|b_1 - b_2|$ is a standard infinitesimal, that is, 0. □

Hyperreals that are not finite are said to be infinite. If c is finite, the unique standard real in the monad of c is called standard part of c and it is denoted by $\mathrm{st}(c)$.

In the following lemma, the expressions on the left can be formalised as first-order sentences, and therefore they hold in $\mathbb{R}$ if and only if they hold in ${}^*\mathbb{R}$.

2.24 Proposition *For every $f : \mathbb{R} \to \mathbb{R}$, for every $a, l \in \mathbb{R}$ the following equivalences hold.*

a. $\lim_{x\to+\infty} fx = +\infty \quad \Leftrightarrow \quad {}^*f(c)$ *is positive and infinite for every infinite $c > 0$.*

b. $\lim_{x\to+\infty} fx = l \quad \Leftrightarrow \quad {}^*f(c) \approx l$ *for every infinite $c > 0$.*

c. $\lim_{x\to a} fx = +\infty \quad \Leftrightarrow \quad {}^*f(c)$ *is positive and infinite for every $c \approx a \neq c$.*

d. $\lim_{x\to a} fx = l \quad \Leftrightarrow \quad {}^*f(c) \approx l$ *for every for every $c \approx a \neq c$.*

Proof We prove part d and we leave the other parts as an exercise. For $\Rightarrow$, assume that the left-hand side of the equivalence d holds and we write it as a first-order sentence:

1. $$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x\, \Big[0 < |x - a| < \delta \quad \to \quad |fx - l| < \varepsilon\Big].$$

By assumption, the formula 1 holds in $\mathbb{R}$, or, equivalently, in ${}^*\mathbb{R}$. In what follows, we use certain abbreviations which we assume the reader can translate into first-order formulas. For consistency with standard notation in analysis, we use the Greek letters ε and δ as variables. The symbols $\dot{\varepsilon}$ e $\dot{\delta}$ denote parameters.

We now check that ${}^*f(c) \approx l$ holds for every $c \approx a \neq c$, that is, $|{}^*fc - l| < \dot{\varepsilon}$ for every positive standard $\dot{\varepsilon}$. Fix a standard positive $\dot{\varepsilon}$ and let $\dot{\delta} \in \mathbb{R}$ be a standard real obtained from 1. By elementarity, we have

$$ {}^*\mathbb{R} \vDash \forall x \left[0 < |x - a| < \dot{\delta} \quad \rightarrow \quad |fx - l| < \dot{\varepsilon}\right], $$

where $\dot{\varepsilon}$ and $\dot{\delta}$ are now parameters. If $a \approx c \neq a$, then $0 < |c - a| < \dot{\delta}$ certainly holds (because $\dot{\delta}$ is standard). Hence 1 gives $|{}^*fc - l| < \dot{\varepsilon}$.

For $\Leftarrow$, assume that 1 is false, that is, assume that, in $\mathbb{R}$,

2. $$ \exists \varepsilon > 0\, \forall \delta > 0\, \exists x \left[0 < |x - a| < \delta \quad \wedge \quad \varepsilon \leq |fx - l|\right], $$

We want to show that ${}^*f(c) \not\approx l$ for some $c \neq a$ infinitely close to a. Fix a witness $\dot{\varepsilon}$ of this formula in $\mathbb{R}$ — so $\dot{\varepsilon}$ is a standard real. Now, by elementarity we have

$$ {}^*\mathbb{R} \vDash \forall \delta > 0\, \exists x \left[0 < |x - a| < \delta \quad \wedge \quad \dot{\varepsilon} \leq |fx - l|\right]. $$

Let $\dot{\delta}$ be an arbitrary infinitesimal. Then

$$ {}^*\mathbb{R} \vDash \exists x \left[0 < |x - a| < \dot{\delta} \quad \wedge \quad \dot{\varepsilon} \leq |fx - l|\right]. $$

Any c that witnesses the truth of this formula in ${}^*\mathbb{R}$ is such that $c \approx a \neq c$ and, simultaneously, $\varepsilon \leq |{}^*fc - l|$. But $\dot{\varepsilon}$ was chosen to be standard, so ${}^*fc \not\approx l$. □

The following corollary is immediate.

2.25 Corollary *For every $f : \mathbb{R} \to \mathbb{R}$ the following are equivalent*

a. *f is continuous;*

b. *${}^*f(a) \approx {}^*f(c)$ for every pair of finite hyperreals such that $c \approx a$.* □

In Corollary 2.25, it is important to restrict c to *finite* hyperreals, otherwise we get a stronger property.

2.26 Proposition *For every $f : \mathbb{R} \to \mathbb{R}$, the following are equivalent*

a. *f is uniformly continuous;*

b. *${}^*f(a) \approx {}^*f(b)$ for every pair of hyperreals such that $a \approx b$.*

Proof We prove a $\Rightarrow$ b. Recall that f is uniformly continuous if

1. $$ \mathbb{R} \vDash \forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \left[|x - y| < \delta \quad \rightarrow \quad |fx - fy| < \varepsilon\right]. $$

Assume a and let $a \approx b$. We wnat to show that $|{}^*f(a) - {}^*f(b)| < \dot{\varepsilon}$ for every positive standard $\dot{\varepsilon}$. Given a standard positive $\dot{\varepsilon}$, let $\dot{\delta}$ be a standard real obtained from the validity of 1 in $\mathbb{R}$. Now elementarity gives

2. $$ {}^*\mathbb{R} \vDash \forall x, y \left[|x - y| < \dot{\delta} \quad \rightarrow \quad |fx - fy| < \dot{\varepsilon}\right]. $$

In particular,

$$ {}^*\mathbb{R} \vDash \quad |a - b| < \dot{\delta} \quad \rightarrow \quad |fa - fb| < \dot{\varepsilon}. $$

Since $a \approx b$, we have $|a - b| < \dot{\delta}$ for any standard $\dot{\delta}$. It follows that $|{}^*f(a) - {}^*f(b)| < \dot{\varepsilon}$.

To prove b $\Rightarrow$ a we negate a

3. $$\mathbb{R} \vDash \exists \varepsilon > 0 \, \forall \delta > 0 \, \exists x, y \, \Big[|x - y| < \delta \quad \wedge \quad \varepsilon \leq |fx - fy| \Big].$$

We want $a \approx b$ such that $\dot{\varepsilon} \leq |{}^*f(a) - {}^*f(b)|$ for some positive standard $\dot{\varepsilon}$. Let $\dot{\varepsilon}$ be a standard real that witnesses the truth of 3 in $\mathbb{R}$. Elementarity gives

$${}^*\mathbb{R} \vDash \forall \delta > 0 \, \exists x, y \, \Big[|x - y| < \delta \quad \wedge \quad \dot{\varepsilon} \leq |fx - fy| \Big].$$

So we can fix an arbitrary infinitesimal $\dot{\delta} > 0$ and get $a, b \in {}^*\mathbb{R}$ such that

$${}^*\mathbb{R} \vDash |a - b| < \dot{\delta} \quad \wedge \quad \dot{\varepsilon} \leq |fa - fb|.$$

Since $\dot{\delta}$ is infinitesimal, we have $a \approx b$, as required. □

The following proposition is an immediate consequence of Proposition 2.24.

2.27 Proposition *For every unary function f and every standard a, the following are equivalent.*

a. *f is differentiable in a. Note that this holds in $\mathbb{R}$ if and only if it holds in ${}^*\mathbb{R}$.*

b. *for every infinitesimal h, the quotient*
$$\frac{f(a) - f(a + h)}{h}$$
is finite and its standard part is independent from h.

2.28 Exercise Prove that is the function $f : \mathbb{R} \to \mathbb{R}$ is injective, then ${}^*fa \in {}^*\mathbb{R} \smallsetminus \mathbb{R}$ for every $a \in {}^*\mathbb{R} \smallsetminus \mathbb{R}$. □

2.29 Exercise Prove that the following conditions are equivalent for every subset $X \subseteq \mathbb{R}$:

1. X is a finite set;
2. ${}^*X = X$. □

2.30 Exercise Prove that the following conditions are equivalent for every subset $X \subseteq \mathbb{R}$:

1. X is an open set in the usual topology on $\mathbb{R}$;
2. $b \approx a \in {}^*X \quad \Rightarrow \quad b \in {}^*X$ for every standard a and arbitrary b. □

2.31 Exercise Prove that the following conditions are equivalent for every subset $X \subseteq \mathbb{R}$:

1. X is a closed set in the usual topology on $\mathbb{R}$;
2. $a \in {}^*X \quad \Rightarrow \quad \operatorname{st} a \in {}^*X$ for every finite a. □

2.32 Exercise Prove that $|\mathbb{R}| \leq |{}^*\mathbb{Q}|$, that is, that the cardinality of ${}^*\mathbb{Q}$ is at least that of the continuum. (Hint: define an injective function $f : \mathbb{R} \to {}^*\mathbb{Q}$ by choosing a nonstandard rational in the monad of every standard real.) □