AROUND FINITE BASIS RESULTS FOR QUASI-ORDERS

Introductory Lecture

During this course, I will take a closer look at some specific quasi-orders on classes of analytic objects or structures.

- continuous homomorphism on analytic graphs and box-open hypergraphs on analytic spaces.
- topological embeddability and continuous reducibility on Borel/analytic functions.

I will first prove how finite basis results on graphs and hypergraphs can yield finite basis results for functions.

Then I will discuss the possibility of guaranteeing finite basis results on functions, that is the possibility and/or impossibility for topological embeddability and cont. reducibility to be well-quasi-ordered.

What is a finite basis result for a quasi-order?

Def. A quasi-order is a transitive and reflexive relation $\leq_{\mathcal{C}}$ on a class $\mathcal{C}$.

Typical example: $\mathcal{C}$ is a class of structures, and $\leq_{\mathcal{C}}$ is given by the existence of a morphism between two of these. So:

- Lin, the class of all linear orders, with embeddings as morphisms (order-preserving injections)

  K, L linear orders

  $K \leq_i L$ if there exist $\sigma : K \to L$ embedding.

- Topological spaces w/ continuous injections

- Many others w/ graphs, functions.

Note right away that a partial order is a qo, but the converse is not true, as shown for instance by Lin: bi-embeddability is <u>not</u> identity, so <u>many</u> linear orders are bi-comparable without being equal.

We systematically denote $\equiv_{\mathcal{C}}$ the equiv. relation associated with $\leq_{\mathcal{C}}$, that is

$$x \equiv_{\mathcal{C}} y \iff x \leq_{\mathcal{C}} y \text{ and } y \leq_{\mathcal{C}} y.$$

Whenever this makes sense we can consider

$\mathcal{C}/_{\equiv_{\mathcal{C}}}$, and $\leq_{\mathcal{C}}$ induces a <u>partial</u> order on these classes.

What is the point in considering a qo then?

The quotient might have lost in structure!

<u>Ex</u> Look at Lin, the countable elements of $\mathcal{L}IN$. They are all representable in $2^{\mathbb{N}\times\mathbb{N}}$ which is a very nice space (it is Polish, see later).
On the other hand $\text{Lin}/_{\leq_i}$ has (as we might see during the course at some point) $\aleph_1$ many elements...

<u>Basis</u> Let $\leq_{\mathcal{C}}$ be a qo on $\mathcal{C}$, $\mathcal{D}$ a subclass of

$\mathcal{C}$. A set $\mathcal{B}$ in $\mathcal{D}$ is a basis for $\mathcal{D}$ if for every element d in $\mathcal{D}$ there is b in $\mathcal{B}$ such that $b \leq_{\mathcal{C}} d$.

A finite basis result is a statement of existence of a finite basis, for a subclass $\mathcal{D}$ of a certain class $\mathcal{C}$, with respect to a given quasi-order.

Example: Let ω denote the usual order on $\mathbb{N}$, and ω^* its reverse

Fact $\{\omega, \omega^*\}$ is a basis wrt embeddability for the infinite linear orders.

Generalising the previous fact to uncountable linear orders is independent of ZFC! But

Thm (Moore) Assuming the Proper Forcing Axiom there is a five-elements basis for uncountable linear orders.

Getting closer to our framework

We work with topogical spaces that have nice properties.

- Polish spaces : completely metrisable spaces.

ex : • $\mathbb{R}$, $\mathbb{R}^n$ for $n \in \mathbb{N}$, $\mathbb{R}^{\mathbb{N}}$, $\mathbb{C}$

- Any separable Banach space
- $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ with the product topology of the discrete one.

$2^{\mathbb{N}}$ is homeomorphic to the usual Cantor space, built either geometrically by cutting in three a segment and throwing away the middle part, or looking at

$$\left\{ x \in [0,1] \mid x = \sum_{i=0}^{\infty} \frac{k}{3^{i+1}},\ k \in \{0,2\} \right\}.$$

. The Baire space: $\mathbb{N}^{\mathbb{N}}$ w/ product topology.

It is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$. Each $s \in \mathbb{N}^{<\mathbb{N}}$ gives a basic open set: $N_s := \{ x \in \mathbb{N}^{\mathbb{N}} \mid s$ is an init. segment of $x \}$
written $s \sqsubseteq x$.

Same topology for $2^{\mathbb{N}}$.

A basis result for Polish space:

<u>Thm</u> (Cantor) $\boxed{\text{For every uncountable Polish space } X \text{ there is a continuous injection } 2^{\mathbb{N}} \longrightarrow X.}$

This is also know as the Perfect Set Property for Polish spaces.

- Analytic spaces. Continuous images of Polish spaces.

  Or equivalently, the empty set and all continuous images of the Baire space.

This allows to talk about a really wide variety of spaces.

Ex: Continuously differentiable functions on $[0,1]$

Given f continuous on $[0,1]$ the set $\{x \in [0,1] \mid f'(x) \text{ exists}\}$

For any Polish X the set of uncountable closed subsets of X.

Thm (Suslin) Analytic spaces have the Perfect Set Property.

- Borel sets. Given a topological space X, the Borel sets in X is the σ-algebra generated by the

open sets, that is the smallest family that contains the open sets and that is closed under cble $\cap$ and comple-
-ments. In general it is messy, but for Polish spaces it looks like this:

$$\begin{array}{ccccccccccccc}
 & \Sigma^0_1 & & & & \Sigma^0_2 & & & & & \Sigma^0_\omega & & \\
\Delta^0_1 \subseteq & & \subseteq & \Delta^0_2 & \subseteq & & \subseteq & \Delta^0_3 & \cdots\cdots & & & \subseteq & \Delta^0_{\omega+1} \cdots\cdots \\
\subseteq & \Pi^0_1 & \subseteq & & \subseteq & \Pi^0_2 & \subseteq & & & & \Pi^0_\omega & \subseteq &
\end{array}$$

Among the examples above the first two are Borel sets, and the last one, provided that X is uncountable itself, is not. (Hurewicz).

analytic is denoted Σ^1_1, coanalytic Π^1_1.

Suslin proved that for Polish, Borel $= \Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$.

Graphs and definable chromatic numbers

<u>Def</u> : • A <u>graph</u> is an irreflexive symmetric relation on a set X, sometimes denoted (X, G).

• A K-coloring of (X, G) for some set K is a map $c : X \to K$ satisfying $(x, y) \in G \Rightarrow c(x) \neq c(y)$

• The <u>chromatic number</u> of G is the minimal size of a set K (or without AC the minimal cardinal K) such that G has a K-coloring.

• G is <u>acyclic</u> if for all injective sequences $(x_i)_{i \leq n}$ for some $n \in \mathbb{N}$, if $(x_n, x_0) \in G$ then either $n = 1$ or there exists $i < n$ with $(x_i, x_{i+1}) \notin G$.

Prop Using the axiom of choice, every acyclic graph has chromatic number 2.

Pf. Suppose first that (X, G) is acyclic and connected. That is $\forall x, y \in X \ \exists (x_i)_{i \leq n}$ st $x_0 = x$, $(x_i, x_{i+1}) \in G$ and $(x_n, y) \in G$, called a <u>path</u> from x to y.

By acyclicity an injective such path is unique, we call its length the distance from x to y, noted $d(x,y)$.

Fix any x in X gives a 2-colouring as follows.

$$\sigma_{x,X} : X \longrightarrow 2$$

$$x \longmapsto 0$$

$$y \neq x \longmapsto \begin{cases} 0 & \text{if } d(x,y) \text{ is even} \end{cases}$$

$\quad\quad\quad\quad 1$ otherwise.

Consider now (Y, G) any acyclic graph.
Call $\mathcal{G}$ the set of connected components of Y, and fix a choice function $\iota : \mathcal{G} \to Y$, $\iota(X) \in X$.
This gives a 2-coloring $\sigma = \bigcup_{X \in \mathcal{G}} \sigma_{\iota(X), X}$. $\square$

This begs for the question: if G is an acyclic and "definable" graph, can we find a "definable" 2-coloring?
Suppose that G is a graph on a topological space X, a K-coloring c of G is <u>Borel</u> if $\forall i \in K$ $c^{-1}(\{i\})$ is Borel.
The <u>Borel chromatic number</u> of G is the minimum card.

κ such that G has a Borel κ-coloring.

The more precise problem becomes:

Take (X, G) an acyclic graph. If X is Polish and G analytic, do we have a Borel 2-coloring of G?

The answer is no!

We are going to define a (family of) counterexample(s) on $2^{\mathbb{N}}$.

<u>Def</u>: Fix $S \subseteq 2^{<\mathbb{N}}$.

- Note $G_S = \bigcup_{s \in S} \left\{ (s^\frown(i)^\frown z, s^\frown(1-i)^\frown z) \,\middle|\, \begin{matrix} i = 0,1 \\ z \in 2^{\mathbb{N}} \end{matrix} \right\}$
- Say that S is <u>sparse</u> if for all $n \in \mathbb{N}$ $2^n \cap S$ is <u>at most</u> a singleton

• Say that S is <u>dense</u> if $\forall t \in 2^{<\mathbb{N}}\ \exists s \in S\ t \sqsubseteq s$.

<u>Prop</u> : $\boxed{\text{If } S \text{ is sparse then } G_S \text{ is acyclic.}}$

<u>Pf</u>. Take an injective $(x_i)_{i \leq n}$ in $2^{\mathbb{N}}$, and suppose that $(x_0, x_n) \in G_S$ and that $n \neq 1$ (so $n > 1$).

We want to find $i < n$ st $(x_i, x_{i+1}) \notin G_S$.

<u>Case 1</u> : if there is $i < n$ st $\{m \in \mathbb{N} \mid x_i(m) \neq x_{i+1}(m)\}$ is of size at least 2 then by definition of G_S for such an i, $(x_i, x_{i+1}) \notin G_S$.

So for all $i < n$ we have $s_i \in 2^{<\mathbb{N}}$, $\varepsilon_i \in 2$ and $z_i \in 2^{\mathbb{N}}$ with

$$x_i = s_i \frown (\varepsilon_i) \frown z_i \quad , \quad x_{i+1} = s_i \frown (1-\varepsilon_i) \frown z_i . \quad (*)$$

<u>Case 2</u> : for some $i < n$ $s_i \notin S$. Then $(x_i, x_{i+1}) \notin G_S$

So suppose that for all $i < n$ $s_i \in S$ (towards contradiction)

Call $s_n \in S$ the witness that $(x_0, x_n) \in G_S$.

Using $(*)$, injectivity of $(x_i)_{i \leq n}$ yields injectivity of $(s_i)_{i \leq n}$. Since S is sparse, we have injectivity of $(m_i)_{i \leq n}$ where $m_i = \lg(s_i)$ the length of s_i.

Using $(*)$ again, we have for all $i < n$

$$j \leq i \Rightarrow x_j(m_i) = \varepsilon_i \quad \text{and} \quad i < j \Rightarrow x_j(m_i) = 1 - \varepsilon_i$$

by injectivity of $(m_i)_{i \leq n}$.

But then $x_0(m_0) \neq x_n(m_0)$ <u>and</u> $x_0(m_n) \neq x_n(m_n)$

Contradiction. $\square$

Prop: If S is dense then G_S admits no

Baire-measurable coloring $c : 2^{\mathbb{N}} \longrightarrow \mathbb{N}$.

Pf: Take $c : 2^{\mathbb{N}} \longrightarrow \mathbb{N}$ Baire-measurable.

Since $2^{\mathbb{N}}$ satisfies the Baire category theorem, there exists $m \in \mathbb{N}$ and $s \in 2^{<\mathbb{N}}$ such that $c^{-1}(\{m\})$ is comeager in N_s. Use density to find $t \in S$ with $s \sqsubseteq t$, note that $c^{-1}(\{m\})$ is also comeager in N_t.

The map $\varphi_t : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$

$$x \longmapsto \begin{cases} t^\frown(1-i)^\frown z & \text{if } x = t^\frown(i)^\frown z \\ x & \text{otherwise} \end{cases}$$

is a homeomorphism, so $\varphi_t(c^{-1}(\{m\}))$ is comeager in N_t as well colouring in N_f as well.

So $N_f \cap c^{-1}(\{m\}) \cap \varphi_f^{-1}(c^{-1}(\{m\})) \neq \emptyset$,

which finally means that $c^{-1}(\{m\})$ contains a G_S-edge, so c is not a coloring of G_S. $\square$