The $\mathbb{G}_0$ - dichotomy

Def: X, Y sets, $n \in \mathbb{N}$, $R \subseteq X^n$, $R' \subseteq Y^n$.

$A \subseteq X$ is R-independent if $A^2 \cap R = \phi$

A map $\varphi : X \to Y$ is:

- A homomorphism from R to R' if $\bar{x} \in R \Rightarrow \varphi^n(\bar{x}) \in R'$
- A reduction from R to R' if $\bar{x} \in R \Leftrightarrow \varphi^n(\bar{x}) \in R'$.

Rmk If φ is a homomorphism from a graph (X, G) to a graph (X', G') then any K-coloring c of G' yields a K-coloring $c \circ \varphi$ of G. So the chromatic number of G' is an upper bound for that of G.

Our objective is to show the $\mathbb{G}_0$ - dichotomy:

Thm (Kechris - Solecki - Todorcevic) Suppose that

X is a Hausdorff space, G an analytic graph on X, and $S \subseteq 2^{<\mathbb{N}}$ a sparse dense set.

Then exactly one of the following holds:

1) G has ctble Borel chromatic number

2) There is a continuous homomorphism $\pi : 2^{\mathbb{N}} \longrightarrow X$ from G_S to G.

We are going to follow Ben Miller's methods. They are purely classical, that is no forcing and/or effective methods are involved. Most of the important dichotomy results in descriptive set theory now have a proof that either uses Miller's technique or is obtained as a corollary of (a variant of) the G_0-dichotomy.

We proceed by steps. Note first that the incompatibility between the two sides of the dichotomy follows from the remark above and the fact (seen in Lecture 1) that if S is dense then G_S has uncountable chromatic number. We then proceed by steps. In what follows, X is <u>always</u> a Hausdorff space, and G <u>always</u> an analytic graph on X. Fix continuous functions $\Phi_G : \mathbb{N}^{\mathbb{N}} \to G$ and $\Phi_X : \mathbb{N}^{\mathbb{N}} \longrightarrow X$ such that Φ_G is surjective, and Φ_X is onto $\mathrm{proj}_0(G) \cup \mathrm{proj}_1(G)$, the union of the two projections of G in X.

<u>Thm</u>: Either G has countable Borel chromatic number, or for all sparse sets $S \subseteq 2^{<\mathbb{N}}$ there is a continuous

homomorphism from G_S to G.

We are first going to prove the weaker version, the one that ignores the words in yellow. We give indications in yellow in the proof on how to obtain the stronger version.

Fix a sparse set $S \subseteq \{s_n \mid n \in \mathbb{N}\}$ such that $|s_n| = n$

We will recursively define a decreasing sequence $(X_\alpha)_{\alpha<\omega_1}$ of Borel subsets of X such that G has a countable Borel coloring outside X_α for all $\alpha < \omega_1$.

Set $X_0 = X$ and $X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha$ if $\lambda < \omega_1$ is limit.

A few preliminaries before defining $X_{\alpha+1}$ from X_α.

- An <u>approximation</u> is $a = (n^a, \phi^a, (t_n^a)_{n<n^a})$

  where $n^a \in \mathbb{N}$, $\phi^a : 2^{n^a} \to \mathbb{N}^{n^a}$, and $t_n^a : 2^{n^a-(n+1)} \to \mathbb{N}^{n^a}$ for $n<n^a$

- An approx. b is a <u>one-step extension</u> of an approx. a if:

  ① $n^b = n^a + 1$
  ② $\forall s \in 2^{n^a}\ \forall t \in 2^{n^b}\ (s \sqsubseteq t \Rightarrow \phi^a(s) \sqsubseteq \phi^b(t))$
  ③ $\forall n<n^a\ \forall s \in 2^{n^a-(n+1)}\ \forall t \in 2^{n^b-(n+1)}\ (s \sqsubseteq t \Rightarrow t_n^a(s) \sqsubseteq t_n^a(t))$.

- A <u>configuration</u> is $\gamma = (n^\gamma, \phi^\gamma, (t_n^\gamma)_{n<n^\gamma})$ where

  $n^\gamma \in \mathbb{N}$, $\phi^\gamma : 2^{n^\gamma} \to \mathbb{N}^{\mathbb{N}}$ and $t_n^\gamma : 2^{n^\gamma-(n+1)} \to \mathbb{N}^{\mathbb{N}}$ for $n<n^\gamma$

  that satisfies $\phi_G \circ t_n^\gamma(t) = (\phi_X \circ \phi^\gamma(s_n{}^\frown(0)^\frown t), \phi_X \circ \phi^\gamma(s_n{}^\frown(1)^\frown t))$

- Config γ is <u>compatible</u> w/ approx. a if

  ⓐ $n^a = n^\gamma$
  ⓑ $\forall s \in 2^{n^a}\quad \phi^a(s) \sqsubseteq \phi^\gamma(s)$

(c) $\forall n < n^a \ \forall s \in 2^{n^a-(n+1)} \ T^a_n(s) \subseteq T^\gamma_n(s)$

- Config. γ is <u>compatible</u> w/ $Y \subseteq X$ if $\Phi_X \circ \Phi^\gamma(2^{n^\gamma}) \subseteq Y$.
- Approx. a is <u>Y-terminal</u> if no config. is compat. w/ <u>both</u> Y <u>and</u> a one-step extension of a.
- $A(a, Y) = \{\Phi_X \circ \Phi^\gamma(s_{n^a}) \mid \gamma \text{ is compat. w/ both } a \text{ and } Y\}$

<u>Lemma 1</u>: Approx a is Y-terminal $\Rightarrow$ $A(a,Y)$ is G-independent

<u>Pf</u>: Proceed by contraposition. Suppose that there are config. γ_0 and γ_1, both compat. w/ a and Y, ST $\left(\Phi_X \circ \Phi^{\gamma_0}(s_{n^a}), \Phi_X \circ \Phi^{\gamma_1}(s_{n^a})\right) \in G$.

Set $d \in \mathbb{N}^{\mathbb{N}}$ ST $= \Phi_G(d)$. Define a config. γ

as follows: $n^\gamma = n^a + 1$, $\phi^\gamma(t^\frown(i)) = \phi^{\gamma_i}(t)$ for all $t \in 2^{n^a}$ and $i < 2$, $T^\gamma_n(t^\frown(i)) = T^{\gamma_i}_n(t)$ for $i<2$, $n < n^a$ and $t \in 2^{n^a-(n+1)}$, and $T^\gamma_{n^a}(\emptyset) = d$.

The unique approx. b compatible w/ γ is a one-step extension of a compatible w/ Y. $\boxtimes$

Finally, define $X_{\alpha+1} = X_\alpha \setminus \bigcup \{A(a, A) \mid a \text{ is } X_\alpha\text{-terminal}\}$

Note that since X_α is Borel, $A(a, X_\alpha)$ is analytic. Now since $A(a, X_\alpha)$ is G-independent, there exists a G-independent Borel $B(a, X_\alpha) \supseteq A(a, X_\alpha)$, by a proposition to come after the proof.

Define $X_{\alpha+1} = X_\alpha \setminus \bigcup \{B(a, X_\alpha) \mid a \text{ is } X_\alpha\text{-terminal}\}$

Now, as $(X_\alpha)_{\alpha<\omega_1}$ is decreasing, the sets of X_α-terminal approx. are <u>increasing</u>. Since there are only countably many possible approximations, there is an $\alpha < \omega_1$ such that the set of X_α-terminal approx. and the set of $X_{\alpha+1}$-approx. are the same. Now if a_0, the only approx. w/ $n^{a_0} = 0$, is X_α-terminal, then as $A(a_0, X_\alpha) = X_\alpha$, fix a map $x \longmapsto (\beta_x, a_x)$ with $x \in X_{\beta_x} \setminus X_{\beta_x+1}$, a_x is X_{β_x}-terminal and [$x \in A(a_x, X_{\beta_x})$] $a \in B(a_x, X_{\beta_x})$. By Lemma 1 and the remark following it, it is a countable Borel coloring.

Otherwise, a_0 is not X_α-terminal, hence not $X_{\alpha+1}$-terminal.

<u>Lemma 2</u>: If a is not $X_{\alpha+1}$-terminal, then there is a one-step extension of a that is not X-terminal

one-step extension of a that is not X_α-terminal.

Pf: Take a one-step extension b of a along with a config. γ that is compatible w/ both b and $X_{\alpha+1}$. Hence $\Phi_X \circ \Phi^\gamma(s^{n^b}) \in X_{\alpha+1}$, so $A(b, X_\alpha) \cap X_{\alpha+1} \neq \emptyset$, and b is not X_α-terminal. $\boxtimes$

Inductively apply Lemma 2 in order to define a sequence $(a_n)_{n\in\mathbb{N}}$ of approximations w/ a_{n+1} a one-step extension of a_n for all $n \in \mathbb{N}$.

For $c \in 2^{\mathbb{N}}$, define $\phi(c) = \bigcup_{n\in\mathbb{N}} \Phi^{a_n}(c)$ and for $n \in \mathbb{N}$, $\Psi_n(c) = \bigcup_{m>n} \Psi_m^{a_m}(c \upharpoonright (m-(n+1)))$ $\quad \phi, \Psi_n : 2^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ are continuous functions. We need only show that $\pi = \Phi_X \circ \phi$

is a homomorphism from G_S to G. We prove that:

$$\forall c \in 2^{\mathbb{N}} \; \forall n \in \mathbb{N} \quad \underbrace{\phi_G \circ t_n(c)}_{\text{take } U \text{ open}} = \underbrace{(\phi_X \circ f(s_n \frown (0) \frown c), \phi_X \circ f(s_n \frown (1) \frown c))}_{\text{take } V \text{ open}}.$$

By Hausdorffness, it is enough to show that $U \cap V \neq \emptyset$.

So use continuity to find $m > n$ so that letting $s = c \upharpoonright m-(n+1)$,

we have $\phi_X(N_{\phi^{a_m}(s_n \frown (0) \frown s)}) \times \phi_X(N_{\phi^{a_m}(s_n \frown (1) \frown s)}) \subseteq U$

and $\phi_G(N_{t_n^{a_m}(s)}) \subseteq V$. Since a_m is not X_α-terminal

there is a configuration γ compatible w/ a_m and X_α,

so $\phi_G \circ t_n^{a_m}(s) \in U \cap V$, as desired. $\square$.

Here is the proposition needed for the yellow version.

Prop If $A \subseteq X$ analytic is G-independent, then there is a G-independent Borel set $B \supseteq A$.

Pf. A analytic disjoint from $\mathrm{proj}_0(G \cap X \times A)$, which is also analytic, so there is a Borel $B_0 \supseteq A$ disjoint from $\mathrm{proj}_0(G \cap X \times A)$. Similarly A is disjoint from the analytic $\mathrm{proj}_1(G \cap B_0 \times X)$, so there is a Borel $B_1 \supseteq A$ disjoint from $\mathrm{proj}_1(G \cap B_0 \times X)$. The Borel set $B = B_0 \cap B_1$ is as desired. $\square$

Some applications

(Proof sketch)

PSP for analytic.: Take $A \subseteq X$ analytic, X Polish.

Consider G_A: $(x,y) \in G_A$ if $x \neq y$ and $(x,y) \in A^2$.
Any G-independent set contains at most one point of A, so a countable coloring of $G_A \Rightarrow A$ is countable.
Otherwise take $\varphi : 2^{\mathbb{N}} \to X$ from G_S to G_A, w/ S dense.
Note that $\varphi \restriction U$ is never constant when U is open, so one can recursively build a continuous $\psi : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ st $\varphi \circ \psi(N_s) \cap \varphi \circ \psi(N_t) = \emptyset$ $\forall s,t \in 2^n$ $\forall n \in \mathbb{N}$. $\square$

Silver's theorem is an application of the weak version.

Recall that a <u>partial transversal</u> of an equiv. relation is a set intersecting every equivalence class in at most one point.

<u>Thm</u> (Silver) X Hausdorff, E co-analytic equiv. relation on X. Then exactly one of the following holds:
1. E has countably many classes.
2. There is a perfect partial transversal of E.

<u>Pf</u> (Sketch): 1 and 2 are mutually exclusive since $=_{2^\omega}$ has uncountably many classes. Set $G = X^2 \setminus E$, G is an analytic graph. If G has a countable coloring, then as any G-independent set is contained in a <u>single</u> E-class, we are in situation 1. Otherwise there is a cont. homom. Φ from G_S to G, with S dense. Now since every $\Phi^{-1}(E)$-class is G_S-independent, it is meager, so by Kuratowski-Ulam $\Phi^{-1}(E)$ itself is meager,

so by Mycielski's Thm it contains $=_{2^{\mathbb{N}}}$. Then so does E. $\square$

Connected to Souslin's hypothesis:

<u>Thm</u> (Friedman, Shelah) X Polish, R co-analytic linear qo on X. Exactly one holds:

1. R is separable
2. there is a perfect subset of $X \times X$ whose defining closed intervals that are pairwise disjoint and w/ $\neq \emptyset$ interior.

E equiv. rel is <u>countable</u> if all its classes are.

$x \mathbb{E}_0 y$ on $2^{\mathbb{N}}$ iff $\exists n \in \mathbb{N}\ \forall m \geq n\ x(m) = y(m)$.

<u>Thm</u> (Glimm, Effros, Jackson-Kechris-Louveau, Weiss)
X Polish, E ctble equiv. relation analytic on X.
Exactly one:

1. X is the union of ctbly many Borel partial transversal of E
2. there is a cont. embedding of $\mathbb{E}_0$ in E.

There are applications...

There are generalisations: to finite-dimensional hypergraphs, to infinite-dimensional hypergraphs MODULO a comeager set, to sequences of graphs.

<u>Thm</u> (Miller?) X Polish vector space ST linear dependance is co-analytic. Then $\forall A \subseteq X$ analytic, exactly one:
① span(A) is of ctble dimension.
② there is a lin. indep. pfct subset of A.

<u>Thm</u> (Harrington-Kechris-Louveau) X Polish, E Borel equiv. rel. on X. Exactly one:
① E Borel reduces to $=_{2^{\mathbb{N}}}$.
② there is a cont. embedding from $\mathbb{E}_0$ to E.

A quasi-order R is lexicographically linearizable if there exists a qo $S \supseteq R$ ST $\equiv_S = \equiv_R$ and S is reducible to a lexicographic order.

Thm (Harrington - Marker - Shelah) X Polish, R Borel qo
Either there is a pfct antichain or R is lexico-
-graphically linearizable.