A variant of $2^{\mathbb{N}}$, Hurewicz's dichotomy, and topological embeddability

In this lecture, we start talking about topological embeddability between functions.
We first need one more consequence of the box-open hypergraph dichotomy: Hurewicz's dichotomy.

Using hypergraphs to extend continuous functions

Given X a topological space, and $A \subseteq X$, note $\mathrm{Cnvg}(A, X)$ the sequences in A converging in X.
Write simply $\mathrm{Cnvg}(X)$ for $\mathrm{Cnvg}(X, X)$.

<u>Prop 1</u>: Let X, Y be metric spaces, and $A \subseteq X$.
Suppose that there exists a homomorphism φ from $\mathrm{Cnvg}(A, X)$ to $\mathrm{Cnvg}(Y)$.
Then φ is continuous and admits a continuous ext-

-ension from $\overline{A}$, the closure of A, to Y.

Pf: We first prove that φ is continuous. Take $x_n \longrightarrow x$ in A. Suppose that $\varphi(x_n) \longrightarrow y \neq \varphi(x)$. Consider the sequence $(y_n)_n$ in A w/ $y_{2n} = x_n$ and $y_{2n+1} = x$. We have $y_n \longrightarrow x$ in A, but $(\varphi(y_n))_n$ does not converge, contradicting the fact that φ is a homomorphism.

For $x \in \overline{A} \setminus A$, take any $(x_n)_n \in \mathrm{Cnvg}(A, X)$ w/ $x_n \longrightarrow x$, and fix $\overline{\varphi}(x) = \lim_{n \in \mathbb{N}} \varphi(x_n)$.

First we prove that $\overline{\varphi}$ is well-defined, that is $\overline{\varphi}(x)$ does not depend on the choice of $(x_n)_n \longrightarrow x$. Suppose, towards contradiction, that there exists $(y_n)_n \longrightarrow x$ in $\mathrm{Cnvg}(A, X)$ ST $\lim \varphi(y_n) = y \neq \overline{\varphi}(x)$. Consider then the sequence $(z_n)_n$ defined by $z_{2n} = x_n$ and $z_{2n+1} = y_n$ for all $n \in \mathbb{N}$. We have both $(z_n)_n \subseteq A$ and $z_n \longrightarrow x$, but $(\varphi(z_n))_n$ fails to converge since $y \neq \overline{\varphi}(x)$. This contradicts the assumption that φ is a homomorphism from $\mathrm{Cnvg}(A, X)$ to $\mathrm{Cnvg}(Y)$.

We finally prove that $\overline{\varphi}$ is continuous. Take $x_n \to x$ in $\overline{A}$, and call $(x_n^0)_{n\in I_0}$ the subsequence of $(x_n)_n$ in A, and $(x_n^1)_{n\in I_1}$ the subsequence in $\overline{A}\setminus A$.

We need to prove that $\overline{\varphi}(x_n) \to \overline{\varphi}(x)$, and it is enough to prove that for $\varepsilon < 2$ if I_ε is ∞ then $\overline{\varphi}(x_n^\varepsilon) \to \overline{\varphi}(x)$. For $\varepsilon = 0$ this follows from the fact that φ is a homom. So suppose that I_1 is infinite. Take a sequence of reals $\varepsilon_n > 0$ ST $\varepsilon_n \to 0$. By definition of $\overline{\varphi}$, for all $n \in \mathbb{N}$ there exists $y_n \in A$ ST both $d_X(x_n^1, y_n) < \varepsilon_n$ and $d_Y(\overline{\varphi}(x_n^1), \varphi(y_n)) < \varepsilon_n$. This implies that $y_n \to x$ and that $\lim \varphi(y_n) = \lim \overline{\varphi}(x_n^1) = \overline{\varphi}(x)$. $\square$

<u>Corollary 2</u>: X, Y metric, $\varphi: A \to Y$ continuous. If φ yields a homomorphism from $\mathrm{Cnvg}(A, X\setminus A)$ to $\mathrm{Cnvg}(Y)$ then φ extends to $\overline{\varphi}$ on $\overline{A}$.

Pf: Continuity of φ implies that $(x_n)_n \mapsto (\varphi(x_n))_n$ also maps $\mathrm{Cnvg}(A)$ to $\mathrm{Cnvg}(Y)$. $\square$

Rmk We have used a criterion for convergence of sequences that we will use regularly in the sequel of the course:

Let $(x_n)_n$ be a sequence in a topological space X, and $(I_k)_{k \leq m}$ a finite covering of $\mathbb{N}$ in infinite sets.

Then $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if for all $k \leq m$ $(x_n)_{n \in I_k}$ converges to x.

A compactification of $\mathbb{N}^{\mathbb{N}}$.

Unsurprisingly it is homeomorphic to $2^{\mathbb{N}}$. It is useful to represent it in a slightly different way, mainly for notational reasons.

We note $\mathbb{N}^{\mathbb{N}}_*$ the set $\mathbb{N}^{\mathbb{N}} \cup \{s^\frown(\infty) \mid s \in \mathbb{N}^{<\mathbb{N}}\}$ equipped w/ the smallest topology making the sets N_s clopen, that is the one generated by the sets N_s and $\mathbb{N}^{\mathbb{N}}_* \setminus N_s$ as subbasis.

<u>Define</u>: • a map $\varphi: \mathbb{N}^{\mathbb{N}}_* \longrightarrow 2^{\mathbb{N}}$, $x \in \mathbb{N}^{\mathbb{N}} \longmapsto \prod_{n\in\mathbb{N}} (0)^{x(n)} {}^\frown (1)$

and $s^\frown(\infty) \longmapsto \left(\prod_{n<|s|} (0)^{s(n)} {}^\frown (1)\right)^\frown (0)^\infty$, and

- Recall the bijections $\mathcal{P}(\mathbb{N}) \longrightarrow 2^{\mathbb{N}}$, $X \longmapsto \mathbb{1}_X$ and $2^{\mathbb{N}} \longrightarrow \mathcal{P}(x)$, $c \longmapsto c^{-1}(\{1\})$

- Call $\Psi : 2^{\mathbb{N}} \longrightarrow \mathbb{N}_*^{\mathbb{N}}$ the following map: if X_c is the increasing enum. of $c^{-1}(\{1\})$ for a given $c \in 2^{\mathbb{N}}$ then $\Psi(c)(n) = X_c(n) - \left(1 + \sum_{m<n} X_c(m)\right)$, and if X_c is finite, we add $\Psi(c)(|X_c|) = \infty$.

$\boxed{\underline{\text{Fact 3}}: \ \varphi \text{ is a homeomorphism from } \mathbb{N}_*^{\mathbb{N}} \text{ to } 2^{\mathbb{N}} \text{ w/ inverse } \Psi.}$

Pf: $\varphi \circ \Psi = id_{2^{\mathbb{N}}}$ and $\Psi \circ \varphi = id_{\mathbb{N}_*^{\mathbb{N}}}$. We show continuity.

Take $t \in 2^{<\mathbb{N}}$, there is a unique $s \in 2^{<\mathbb{N}}$ and $n < lg(t)$ st $s(lg(s)) = 1$ and $t = s^\frown(0)^n$. Set $c = s^\frown(0)^\infty$, $m = |X_c|$ and $u = \Psi(c) \restriction m$, then we have
$$\varphi^{-1}(N_t) = N_u \setminus \bigcup_{i<n} N_{u^\frown(i)} \text{ is open.}$$

Take now $t \in \mathbb{N}^{<\mathbb{N}}$, call $X_t = \left\{ \sum_{i \leq m} t(i) \mid m \in [1, lg(t)] \right\}$ and $s \in 2^{\max(X_t)+1}$ the (finite) characteristic function of X_t then $\Psi^{-1}(N_t) = N_s$. $\square$

When $(x_n)_{n \in \mathbb{N}}$ is a sequence of any kind of objects, and $I \subseteq \mathbb{N}$ we denote by $(x_n)_I$ the sequence generated by I, i.e. $(x_n)_{n \in I}$.

Recall that [illegible]

Recall that $\mathbb{H}_{\mathbb{N}^\mathbb{N}} = \bigcup_{s \in \mathbb{N}^{<\mathbb{N}}} \prod_{n \in \mathbb{N}} N_{s^\frown(n)} \subseteq (\mathbb{N}^{\mathbb{N}})$.

Prop 4: X metric, φ cont. homom. from $\mathbb{H}_{\mathbb{N}^\mathbb{N}}$ to $\mathrm{Cnvg}(X)$.
Then φ extends to a continuous $\overline{\varphi} : \mathbb{N}^{\mathbb{N}}_* \longrightarrow X$.

Pf: We apply Cor. 2: φ is continuous: $\mathbb{N}^\mathbb{N} \longrightarrow X$, and it yields a homomorphism from $\mathrm{Cnvg}(\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})$ to $\mathrm{Cnvg}(X)$.

Take indeed $(x_n)_n \in \mathrm{Cnvg}(\mathbb{N}^\mathbb{N}, \mathbb{N}^\mathbb{N}_* \setminus \mathbb{N}^\mathbb{N})$, we have $x_n \longrightarrow s^\frown(\infty)$ for some $s \in \mathbb{N}^{<\mathbb{N}}$. Any infinite subsequence $(x_n)_I$ contains a further infinite subsequence $(x_n)_J$ for $J \subseteq I \subseteq \mathbb{N}$ that is moreover a subsequence of a $\mathbb{H}_{\mathbb{N}^\mathbb{N}}$-hyperedge contained $\prod_n N_{s^\frown(n)}$ so $(\varphi(x_n))_J$ converges to x_J in X. The same interlacing technique as in Prop 1 yields that if $J \neq J'$ are subsequences of two hyper-edges in $\prod_n N_{s^\frown(n)}$ then $x_J = x_{J'} =: x$, so in the end any subsequences of $(\varphi(x_n))_n$ has a subseq. cvging to x, which means $(\varphi(x_n))_n$ cvges to x. $\square$

Rmk: we used here yet another convergence criterion: The sequence $(x_n)_{\mathbb{N}}$ cvges to x if and only if for all infinite $I \subseteq \mathbb{N}$ there is an ∞ $J \subseteq I$ w/ $(x_n)_J \longrightarrow x$.

Hurewicz's dichotomy

Thm (Hurewicz, Kechris-Louveau-Woodin) X metric, $A \subseteq X$ analytic. Exactly one holds:

1. A is Σ^0_2, or F_σ
2. There is a cont. reduction from $\mathbb{N}^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}_*$ to $A \subseteq X$.

Pf: $\mathbb{N}^{\mathbb{N}}$ is not Σ^0_2 in $\mathbb{N}^{\mathbb{N}}_*$, otherwise by comeagerness one of the closed sets would have non empty interior. So the conditions are incompatible.

Consider the graph $H = \mathrm{Cnvg}(A, X \setminus A)$. Any H-independent sets is a closed subset of A, so if H has a ctble coloring then A is F_σ.

But we have seen that $\mathbb{H}$ is box-open, so the box-open hypergraph dichotomy gives when H has no $\aleph_0$ coloring and cont. homom. φ from $\mathbb{H}_{\omega^\omega}$ to H. Conclude using Prop. 4. $\square$

<u>Rmk</u>: The game characterization seen during Lecture 3 gives the generalisations of the Hurewicz dichotomy to all subsets $A \subseteq X$ under AD.

Topological embeddability between functions

<u>Def</u>: • A topological embedding between spaces X and Y is a continuous map $\sigma : X \longrightarrow Y$ that is injective and st $\sigma^{-1} : \operatorname{im} \sigma \longrightarrow X$ is <u>also</u> continuous.

• When X and Y are metrizable, $\sigma : X \to Y$ is a topological embedding iff for all sequences (x_n)

a topological embedding iff for all sequences $(x_n)_n$ in X, we have $(x_n) \longrightarrow x$ iff $\sigma(x_n) \longrightarrow \sigma(x)$.
In other words, σ is a reduction from $\mathrm{Cnvg}(X)$ to $\mathrm{Cnvg}(Y)$.

$$\begin{tikzcd} Y \arrow[r, "\tau"] & Y' \\ X \arrow[u, "f"] \arrow[r, hook, "\sigma"'] \arrow[ur, phantom, "\circlearrowright"] & X' \arrow[u, "g"'] \end{tikzcd}$$

- Given $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$ we say that f (topologically) embeds in g and we write $f \sqsubseteq g$ if there are two top. emb. $\sigma : X \longrightarrow X'$ and $\tau : \operatorname{im} f \longrightarrow \operatorname{im} g$ st $\tau \circ f = g \circ \sigma$. We also say that (σ, τ) is an (top.) embedding from f to g.

Rmks.
- τ is entirely determined by f, g and σ through the equation $\tau(f(x)) = g(\sigma(x))$.
So if (σ, τ) and (σ, τ') embed f in g then $\tau = \tau'$.
We sometimes say that σ embeds f in g.

- In order for σ to embed f in g, it has to reduce the eq. relation induced by f to that induced by g, so $(f(x) = f(y) \Longleftrightarrow g(\sigma(x)) = g(\sigma(y)))$.
In that case $f(x) \longmapsto g\sigma(x)$ defines an injective function. For σ to embed f in g, it

is necessary and sufficient that this is moreover an embedding.

- If f is injective and g is not then they are incomp. for $\sqsubseteq$.

- If f embeds in g and g is Σ^0_α-measurable, then so is f.

- $f \sqsubseteq g \Rightarrow \mathrm{graph}(f)$ embeds in g, but the converse is false! Think about id and a constant on $\mathbb{N}^{\mathbb{N}}$ or $\mathbb{R}$

First basic results, maximality.

Let $d_0 : \omega+1 \to 2$ be $\mathbb{1}_{\{\omega\}}$, and $d_1 : \omega+1 \to \omega$ a bijection.

$\boxed{\text{Prop 5: } X, Y \text{ sep. metriz.}, f : X \to Y \text{ discontinuous. Then } d_0 \sqsubseteq f \text{ or } d_1 \sqsubseteq f.}$

Pf: Take $x_n \to x$ in X ST $f(x_n) \not\to f(x)$ in Y. By passing to a subseq. we can suppose that $\forall n \in \mathbb{N}$ $x_n \neq x$, and to an even further subseq we can suppose that $(x_n)_n$ is injective in X.

Case 1 $\{f(x_n) \mid n \in \mathbb{N}\}$ finite. Pass to a subseq. on which f is constant and $d_0 \sqsubseteq f$.

Case 2 : $\{f(x_n) \mid n \in \mathbb{N}\}$ infinite. Pass to a subseq. on which f is 1-1 and $d_1 \sqsubseteq f$. $\square$

[CPZ article, Prop. 6.2]

Given a space X denote c_X a constant on X and id_X the identity on X.

Prop6: X, Y Polish spaces, X uncountable, and $f: X \to Y$ Baire-meas. Then $c_{2^{\mathbb{N}}} \sqsubseteq f$ or $id_{2^{\mathbb{N}}} \sqsubseteq f$.

Pf: By Baire measurability there is a comeager Π^0_2-set C ST $f \upharpoonright C$ is continuous, so wlog f is continuous.

If $f^{-1}(\{x\})$ is uncountable for some $x \in Y$, then by embedding $2^{\mathbb{N}}$ in $f^{-1}(\{x\})$ (using PSP), we embed

$C_{2^{\mathbb{N}}}$ in f.

Otherwise f is countable-to-1, so by Lusin-Novikov [see for instance B. Miller's notes, thm 2.2.23]. There is a Borel cover $(B_n)_{n\in\mathbb{N}}$ of X st $\forall n\in\mathbb{N}$ $f\restriction B_n$ is injective. For some $n\in\mathbb{N}$ B_n is uncountable, so there exists an embedd. $\sigma: 2^{\mathbb{N}} \longrightarrow B_n$. But $f\sigma$ being injective and cont. on $2^{\mathbb{N}}$ compact, it is an embedding, so $(f\sigma, \sigma)$ embeds $\mathrm{id}_{2^{\mathbb{N}}}$ into f. $\square$

Prop 7: X, Y Polish, $f: X \to Y$ Borel.
If $\mathrm{im}\, f$ is uncountable then $\mathrm{id}_{2^{\mathbb{N}}} \sqsubseteq f$.

Pf: Same idea, but use first Silver's dichotomy on E_f: $x E_f y$ if $f(x) = f(y)$ to get $C_0 \cong 2^{\mathbb{N}}$ w/ $f\restriction C_0$ injective, then $2^{\mathbb{N}} \cong C_1 \subseteq C_0$ st $f\restriction C_1$ is moreover continuous. Finish as in Prop 7. $\square$

Prop 8: $\mathrm{proj}: [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \longrightarrow [0,1]^{\mathbb{N}}$ is maximal for all continuous functions between separable metrizable spaces wrt top. emb.

Pf : Take X, Y sep. metriz., $f : X \to Y$ continuous.
Universality of $[0,1]^{\mathbb{N}}$ gives embeddings
$\sigma' : X \to [0,1]^{\mathbb{N}}$ and $\tau : Y \to [0,1]^{\mathbb{N}}$
Set
$$\sigma : X \longrightarrow [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$$
$$x \longmapsto (\sigma'(x), \tau f(x))$$
, it is an embedding, and (σ, τ) embeds f in proj. $\square$

Cor 9 $\{\text{proj}, d_0, d_1\}$ is a maximal antichain for functions between sep. metriz spaces wrt $\sqsubseteq$.

Thm (Carroy-Pequignot-Vidnyánszky) X, Y Polish, X unctble, $1 \leq \xi < \omega_1$.
There is no $\sqsubseteq$-maximal Σ^0_ξ-meas. function.

Some questions and problems:

1. What is the most general setting for Prop 5 to 8?
2. Adapt Prop 6 to countable spaces.
3. What becomes the previous Thm for ctble domains?
4. Aside the one given by Cor. 9, other examples of finite maximal $\sqsubseteq$-antichains?

of finite maximal $\subseteq$-antichains?

Meet-embeddings on $\mathbb{N}^{<\mathbb{N}}$ vs endomorphisms of $\mathbb{N}_*^{\mathbb{N}}$.

The meet of seq. $s, t \in \mathbb{N}^{<\mathbb{N}}$ is the seq. $r = s \wedge t$ of max. length st $r \subseteq s$ and $r \subseteq t$.

A meet-emb. is an inj. $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ st $\pi(s \wedge t) = \pi(s) \wedge \pi(t)$.

Prop 10: $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-emb. iff:

1. $\forall i \in \mathbb{N}\ \forall t \in \mathbb{N}^{<\mathbb{N}}\ \pi(t) \subsetneq \pi(t^\frown(i))$
2. $\forall i, j \in \mathbb{N}\ \forall t \in \mathbb{N}^{<\mathbb{N}}\ \big(i \neq j \Rightarrow$
$$\pi(t^\frown(i))(\lg(\pi(t))) \neq \pi(t^\frown(j))(\lg(\pi(t)))\big)$$

Pf: Supp. π is $\wedge$-emb.

1. $\pi(t) = \pi(t) \wedge \pi(t^\frown(i)) \Rightarrow \pi(t) \subsetneq \pi(t^\frown(i))$.
2. $\pi(t) = \pi(t^\frown(i)) \wedge \pi(t^\frown(j))$ and $\pi(t^\frown(i)) \neq \pi(t^\frown(j))$.

Supp. 1 and 2.

Take $s \neq t$. set $r = s \wedge t$. wlog $|r| < |s| \Rightarrow \pi(r) \subsetneq \pi(s)$

so either $r = t$ or $|r| < |t|$ and 2 gives $\pi(s)(|\pi(r)|) \neq \pi(t)(|\pi(r)|)$

In both cases we have $\pi(s) \neq \pi(t)$ and $\pi(r) = \pi(s) \wedge \pi(t)$. $\square$

$\underline{\text{Cor}}$: $\boxed{\text{If } \pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}} \text{ satisfies } \pi(t)^\frown(i) \sqsubseteq \pi(t^\frown(i)) \text{ then it is a } \wedge\text{-embedding.}}$

$\underline{\text{Cor}}$: $\boxed{\text{Any } \wedge\text{-embedd. } \pi \text{ induces a top. emb. } \overline{\pi} : \mathbb{N}^{\mathbb{N}}_* \to \mathbb{N}^{\mathbb{N}}_*}$

Pf : Apply Prop 1, noticing that $(s_n{}^\frown(\infty))_n \to x$ in $\mathbb{N}^{\mathbb{N}}$ then by ① $(\pi(s_n)^\frown(\infty))_n$ cvges in $\mathbb{N}^{\mathbb{N}}$ and $s_n{}^\frown(\infty) \to s^\frown(\infty)$ in $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$ then $\pi(s_n)^\frown(\infty)$ cvges to $\pi(s)^\frown(\infty)$ using ②.

So $s^\frown(\infty) \longmapsto \pi(s)^\frown(\infty)$ is cont. and extends to a cont (and inj) map $\overline{\pi} : \mathbb{N}^{\mathbb{N}}_* \to \mathbb{N}^{\mathbb{N}}_*$ $\square$