A $\sqsubseteq$-basis for non Baire class one functions

To obtain a 6-element basis for non-Baire class one functions, we will use the space $\mathbb{N}^{\mathbb{N}}_*$ (defined in Lecture 4) considered as a union: $\mathbb{N}^{\mathbb{N}} \cup (\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$. We are going to prove a basis result for each of those two pieces, and combine them.

We first need a simple and combinatorial way to define top. emb. from $\mathbb{N}^{\mathbb{N}}_*$ to itself.

Meet-embeddings on $\mathbb{N}^{<\mathbb{N}}$ vs endomorphisms of $\mathbb{N}^{\mathbb{N}}_*$.

The meet of seq. $s, t \in \mathbb{N}^{<\mathbb{N}}$ is the seq. $r = s \wedge t$ of max. length st $r \sqsubseteq s$ and $r \sqsubseteq t$.

A meet-emb. is an inj. $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ st $\pi(s \wedge t) = \pi(s) \wedge \pi(t)$.

Prop 1: $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ is a $\wedge$-emb. iff:

1. $\forall i \in \mathbb{N}\ \forall t \in \mathbb{N}^{<\mathbb{N}}\ \pi(t) \sqsubsetneq \pi(t^\frown(i))$
2. $\forall i, j \in \mathbb{N}\ \forall t \in \mathbb{N}^{<\mathbb{N}}\ \big(i \neq j \Rightarrow$
$$\pi(t^\frown(i))(\mathrm{lg}(\pi(t))) \neq \pi(t^\frown(j))(\mathrm{lg}(\pi(t)))\big)$$

Pf: Supp. π is $\wedge$-emb.

① $\pi(t) = \pi(t) \wedge \pi(t^\frown(i)) \Rightarrow \pi(t) \subsetneq \pi(t^\frown(i))$.

② $\pi(t) = \pi(t^\frown(i)) \wedge \pi(t^\frown(j))$ and $\pi(t^\frown(i)) \neq \pi(t^\frown(j))$.

Supp. ① and ②.

Take $s \neq t$. set $r = s \wedge t$. wlog $|r| < |s| \Rightarrow \pi(r) \subsetneq \pi(s)$

so either $r = t$ or $|r| < |t|$ and ② gives $\pi(s)(|\pi(r)|) \neq \pi(t)(|\pi(r)|)$

In both cases we have $\pi(s) \neq \pi(t)$ and $\pi(r) = \pi(s) \wedge \pi(t)$. □

Cor 2: If $\pi : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}}$ satisfies $\pi(t)^\frown(i) \sqsubseteq \pi(t^\frown(i))$ then it is a $\wedge$-embedding.

This corollary gives a very easy way of building recursively a $\wedge$-emb. whose image is contained in a set that is dense below some $s \in \mathbb{N}^{<\mathbb{N}}$.

Cor 3: Any $\wedge$-embedd. π induces a top. emb. $\overline{\pi} : \mathbb{N}^{\mathbb{N}}_* \to \mathbb{N}^{\mathbb{N}}_*$

Pf: We apply the hypergraph characterisation of continuity seen in Lecture 4, observing that if $(s_n{}^\frown(\infty))_n \to x$ in $\mathbb{N}^{\mathbb{N}}$ then by ① $(\pi(s_n)^\frown(\infty))_n$ cvges in $\mathbb{N}^{\mathbb{N}}$ and $s_n{}^\frown(\infty) \to s^\frown(\infty)$ in $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$

then $\pi(s_n)^\frown(\infty)$ cvges to $\pi(s)^\frown(\infty)$ using (2).

So $s^\frown(\infty) \longmapsto \pi(s)^\frown(\infty)$ is contr. and extends to a contr (and inj) map $\pi : \mathbb{N}_*^{\mathbb{N}} \longrightarrow \mathbb{N}_*^{\mathbb{N}}$ $\square$

Meet-emb. admit a form of diagonal composition:

Given $\pi_0, \ldots \pi_n$, denote $\circ_{i \leq n} \pi_i = \pi_0 \circ \ldots \circ \pi_n$.

$$\boxed{\begin{array}{l}\text{Prop 4 (diagonal composition): Suppose that } (\pi_t)_{t \in \mathbb{N}^{<\mathbb{N}}} \text{ is a sequence of } \wedge\text{-embeddings w/ the property that}\\ \text{for all } t \in \mathbb{N}^{<\mathbb{N}} \quad \pi_t(\mathbb{N}^{<\mathbb{N}}) \subseteq N_t.\\ \text{Then the map } \pi : t \longmapsto \circ_{i \leq |t|} \pi_{t \restriction i} \text{ is a } \wedge\text{-emb.}\end{array}}$$

Pf: Note that for all $i \in \mathbb{N}$ and $t \in \mathbb{N}^{<\mathbb{N}}$ $t^\frown(i) \sqsubseteq \pi_{t^\frown(i)}(t^\frown(i))$, so by prop. 1, composing by $(\circ_{i \leq |t|} \pi_{t \restriction i})$ we get

$\pi(t) \sqsubseteq (\circ_{h \leq |t|} \pi_{t \restriction h})(t^\frown(i)) \sqsubseteq \pi(t^\frown(i))$.

Similarly, $i \neq j \Rightarrow (\circ_{h \leq |t|} \pi_{t \restriction h})(t^\frown(i))(|\pi(t)|)$

$\neq (\circ_{h \leq |t|} \pi_{t \restriction h})(t^\frown(j))(|\pi(t)|)$

which in turn gives $\pi(t^\frown(i))(|\pi(t)|) \neq \pi(t^\frown(j))(|\pi(t)|)$.

So by Prop 1 we are done. $\square$.

Meet-embeddings enjoy a form of Ramsey property.

Prop 4 $\boxed{\text{Sp. } T \subseteq \mathbb{N}^{<\mathbb{N}}. \text{ Then there is a } \wedge\text{-emb } \pi: \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}^{<\mathbb{N}} \text{ st } \pi(\mathbb{N}^{<\mathbb{N}}) \subseteq T \text{ or } \pi(\mathbb{N}^{<\mathbb{N}}) \cap T = \emptyset.}$

Pf: Fix $S \in \{T, \mathbb{N}^{<\mathbb{N}} \setminus T\}$ and $s \in \mathbb{N}^{<\mathbb{N}}$ st S is dense below s. Then recursively build $\pi: \mathbb{N}^{<\mathbb{N}} \to N_s \cap S$ st $\pi(t)^\frown(i) \sqsubseteq \pi(t^\frown(i))$ for all $t \in \mathbb{N}^{<\mathbb{N}}$ and $i \in \mathbb{N}$. $\square$

Prop 5: $\boxed{C \subseteq \mathbb{N}^{\mathbb{N}} \text{ non-meager w/ the Baire property. There is a } \wedge\text{-emb. } \pi \text{ w/ } \pi(\mathbb{N}^{\mathbb{N}}) \subseteq C.}$

Pf. Fix $s \in \mathbb{N}^{<\mathbb{N}}$ st C comeag. in $N_s \cap \mathbb{N}^{\mathbb{N}}$, and dense open U_n in $N_s \cap \mathbb{N}^{\mathbb{N}}$ w/ $\bigcap U_n \subseteq C$. Set $T_n = \{t \in \mathbb{N}^{<\mathbb{N}} \mid N_t \cap \mathbb{N}^{\mathbb{N}} \subseteq U_n\}$ for all $n \in \mathbb{N}$, and build recursively a map $\pi: \mathbb{N}^{<\mathbb{N}} \to N_s \cap \mathbb{N}^{<\mathbb{N}}$ st $\pi(\mathbb{N}^n) \subseteq T_n$, and $\pi(t)^\frown(i) \sqsubseteq \pi(t^\frown(i))$. $\square$

A basis for Baire-meas. fct on $\mathbb{N}^{\mathbb{N}}$, but w/ $\wedge$-emb.

Prop 6. X metric, $\Phi: \mathbb{N}^{\mathbb{N}} \to X$ continuous and nowhere constant.

Then there is a $\wedge$-emb. π ST :

$$\forall i \in \mathbb{N}\ \forall t \in \mathbb{N}^{<\mathbb{N}}\ \overline{\phi(N_{\pi(t^\frown(i))})} \cap \overline{\bigcup_{j \neq i} \phi(N_{\pi(t^\frown(j))})} = \emptyset.$$

Pf Note that each $\phi(N_t)$ is infinite.

Claim: For all $t \in \mathbb{N}^{<\mathbb{N}}$, there is a function $\ell_t : \mathbb{N} \to \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}$ ST $(\ell_t(i)(0))_{i \in \mathbb{N}}$ is 1-1 and the closures of $\phi(N_{t^\frown \ell_t(i)})$ and $\bigcup_{j \neq i} \phi(N_{t^\frown \ell_t(j)})$ are disjoint $\forall i \in \mathbb{N}$.

Subpf: Fix first $b_i \in \mathbb{N}^{\mathbb{N}}$, $b_i \sqsupset t^\frown(i)$ ST $\forall i\ b_i \notin \{b_j \mid j < i\}$. This takes care of inj. Fix then $(a_i)_I \subseteq (b_i)_{\mathbb{N}}$ infinite ST $\{\phi(a_i) \mid i \in I\}$ is discrete.
For all $i \in \mathbb{N}$ pick $\varepsilon_i > 0$ ST $\phi(a_j) \notin B_X(\phi(a_i), \varepsilon_i)$ if $j \neq i$ and $\ell_t(i)$ ST $t^\frown \ell_t(i) \sqsubseteq a_i$ and

$$\phi(N_{t^\frown \ell_t(i)}) \subseteq B_X(\phi(a_i), \varepsilon_i/3)$$

They are as desired, otherwise take $i \in \mathbb{N}$, $j \neq i$, $x \in \phi(N_{t^\frown \ell_t(i)})$, $y \in \phi(N_{t^\frown \ell_t(j)})$ and $d(x,y) \leq \varepsilon_i/3$, and get

$$\begin{aligned} d_X(\phi(a_i), \phi(a_j)) &\leq d_X(\phi(a_i), x) + d_X(x,y) + d_X(y, \phi(a_j)) \\ &\leq \varepsilon_i/3 + \varepsilon_i/3 + \varepsilon_j/3 \\ &\leq \max(\varepsilon_i, \varepsilon_j) \end{aligned}$$

contradiction. ⊠

$= \max(c_i, c_j)$, a contradiction.

Finally define $\pi : \mathbb{N}^{<\mathbb{N}} \longrightarrow \mathbb{N}^{<\mathbb{N}}$ setting $\pi(\emptyset) = \emptyset$ and
$\pi(t^\frown(i)) = \pi(t)^\frown \ell_{\pi(t)}(i) \quad \forall i \in \mathbb{N} \ \forall t \in \mathbb{N}^{<\mathbb{N}}$. □

Thm (Carroy-Miller) X separable metric, Φ Baire-meas.
Then there is a 1-emb $\pi : \mathbb{N}^{<\mathbb{N}} \longrightarrow \mathbb{N}^{<\mathbb{N}}$ ST
$\Phi \circ \pi$ is either constant or a top. embedding.

Pf. Take C comeager ST $\Phi \upharpoonright C$ is continuous, and π_0 w/ $\pi_0(\mathbb{N}^{\mathbb{N}}) \subseteq C$ by Prop 5. If there is $s \in \mathbb{N}^{<\mathbb{N}}$ ST $\Phi \upharpoonright C \cap N_s$ constant then pick π ST $\pi(\mathbb{N}^{\mathbb{N}}) \subseteq C \cap N_s$. Otherwise, use Prop 6 on $\Phi \circ \pi_0$ to get π. □

Functions on $\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}$

This space is countable and dense in itself. Our goal is to show that any function into a metric space can be precomposed to get either a constant, an injection in a discrete space, or a top. emb.

We fix X a metric space. A set is ε-discrete if all distinct points have distance $\geq \varepsilon$. In this section ϕ is always a map from $\mathbb{N}^{\mathbb{N}}_{*} \setminus \mathbb{N}^{\mathbb{N}}$ to X. We also fix an enumeration $(t_n)_n$ of $\mathbb{N}^{<\mathbb{N}}$ st $t_m \sqsubset t_n \Rightarrow m < n$ for all $m, n \in \mathbb{N}$.

Prop 7. $\boxed{\text{Let } \varepsilon > 0 \text{ and } t \in \mathbb{N}^{<\mathbb{N}}. \text{ There is a } \wedge\text{-emb } \pi : \mathbb{N}^{<\mathbb{N}} \to N_t \cap \mathbb{N}^{<\mathbb{N}} \text{ st } \phi \circ \pi \text{ is an inj. into an } \varepsilon\text{-discrete set, or } \phi \circ \pi(\mathbb{N}^{\mathbb{N}}_{*} \setminus \mathbb{N}^{\mathbb{N}}) \subseteq B(x, \varepsilon) \text{ for some } x \in \phi(N_t)}$

Pf. If there is $u \sqsupseteq t$ and F finite w/ $\phi(N_u) \subseteq B(F, \varepsilon)$, then there is $x \in \phi(\mathbb{N}^{\mathbb{N}}_{*} \setminus \mathbb{N})$ st $\{s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s^\frown(\infty)) \in B(x, \varepsilon)\}$ is dense below some $v \sqsupseteq u$, then use a recursive construction.

Otherwise, recursively build $\pi : \mathbb{N}^{<\mathbb{N}} \to N_t \cap \mathbb{N}^{<\mathbb{N}}$ st $\phi(\pi(t_n)^\frown(\infty)) \notin B(\{\phi(t_m)^\frown(\infty) \mid m < n\}, \varepsilon)$ and $\pi(t'_n)^\frown(n) \sqsubseteq \pi(t_n)$ for all $n > 0$, where t'_n is the predecessor of t_n. □

Cor 8: $\boxed{\text{There is a } \wedge\text{-emb. } \pi \text{ st } \phi \circ \pi \text{ is constant or injective.}}$

Pf. Apply Prop 7 w/ an ε-discrete metric on X □

Prop 9. There is a $\wedge$-emb. π st $\phi \circ \pi$ is an injection into an ε-discrete set for some $\varepsilon > 0$ or $\operatorname{diam}(\phi \circ \pi)(N_t) \longrightarrow 0$.

Pf. Suppose that for no $\varepsilon > 0$ there is a $\wedge$-emb π st $\phi \circ \pi$ is an inj. into an ε-discrete set. Fix $(\varepsilon_t)_{t \in \mathbb{N}^{<\mathbb{N}}}$ seq. of positive reals st $\varepsilon_t \longrightarrow 0$, and recursively apply Prop 7 to the functions $\phi_t = \phi \circ (\bigcirc_{n<|t|} \overline{\pi_{t \upharpoonright n}})$ to obtain $\wedge$-emb. $\pi_t : \mathbb{N}^{<\mathbb{N}} \longrightarrow N_t \cap \mathbb{N}^{<\mathbb{N}}$ st $(\phi \circ (\bigcirc_{n \leq |t|} \overline{\pi_{t \upharpoonright n}}))(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}})$ is contained in an ε_t-ball for all $t \in \mathbb{N}^{<\mathbb{N}}$.

The diagonal composition π is the desired $\wedge$-emb. $\square$

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Prop 10. There is a $\wedge$-emb. π st for all $t \in \mathbb{N}^{<\mathbb{N}}$ $((\phi \circ \overline{\pi})(t \frown (i, \infty)))_{i \in \mathbb{N}}$ is convergent, or $\{\phi \circ \overline{\pi}(t \frown (i, \infty)) \mid i \in \mathbb{N}\}$ is closed and discrete.

Pf. For $t \in \mathbb{N}^{<\mathbb{N}}$ take an injection $\ldots : \mathbb{N} \to \mathbb{N}$ st the family

… an injection x_t … the family

$\phi \circ \pi (t^\frown (x_t(i), \infty))_i$ satisfies the required ppty, and set $\pi(\emptyset) = \emptyset$ and $\pi(t^\frown(i)) = \pi(t)^\frown x_{\pi(t)}(i)$. $\square$

<u>Prop 11</u>. $F \subseteq X$ finite, $t \in \mathbb{N}^{<\mathbb{N}}$. There is a $\wedge$-emb π ranging in N_t st either $((\phi \circ \pi)(u^\frown(\infty)))_{u \in \mathbb{N}^{<\mathbb{N}}}$ converges to an element of F or the closure of $(\phi \circ \pi)(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) \cap F = \emptyset$.

<u>Pf</u>: If $S_\varepsilon = \{ s \in \mathbb{N}^{<\mathbb{N}} \mid \phi(s^\frown(\infty)) \in B(F, \varepsilon) \}$ is dense below t $\forall \varepsilon > 0$, then there is $u \sqsupseteq t$ and $x \in F$ st $S_{\varepsilon, x}$ is dense below u for all $\varepsilon > 0$. Fix $(\varepsilon_v)_{v \in \mathbb{N}^{<\mathbb{N}}} \longrightarrow 0$, $\varepsilon_v > 0$, and recursively build $\pi : \mathbb{N}^{<\mathbb{N}} \longrightarrow N_u \cap \mathbb{N}^{<\mathbb{N}}$ w/ $\pi(v) \in S_{\varepsilon_v, x}$ for all $v \in \mathbb{N}^{<\mathbb{N}}$, and $\pi(v)^\frown(i) \sqsubseteq \pi(v^\frown(i))$. Observe that $\phi \circ \pi(v^\frown(\infty)) \longrightarrow x$.

Otherwise, fix $\varepsilon > 0$ and $u \sqsupseteq t$ w/ $N_u \cap S_\varepsilon = \emptyset$ and set $\pi(v) = u^\frown v$. $(\phi \circ \pi)(\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}) \cap F = \emptyset$. $\square$

<u>Prop 12</u>: There is a $\wedge$-emb π w/ $(\phi \circ \pi (t^\frown(\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ cvges or $\forall m < n$ $(\phi \circ \pi)(t_m^\frown(\infty)) \notin \overline{(\phi \circ \pi)(N_{t_n})}$.

Pf Sp that for no $\wedge$-emb. is $((\phi \circ \pi)(t^\frown(\infty)))_{t \in \mathbb{N}^{<\mathbb{N}}}$ cvgt. Recursively apply Prop 11 to the functions

$\phi_t = \phi \circ \left(\circ_{k<|t|} \overline{\pi_{t\restriction k}}\right)$ to obtain $\pi_t : {}^{\uparrow}\mathbb{N}^{<\mathbb{N}} \longrightarrow {}^{\uparrow}\mathbb{N}^{<\mathbb{N}} \cap N_t$
— here $t = t_n$ —
ST $\forall m < n$ $\phi \circ \left(\circ_{k \le |t_m|} \overline{\pi_{t_m \restriction k}}\right)(t_m \frown (\infty))$

$$\notin \overline{\left(\phi \circ \left(\circ_{k \le |t_n|} \overline{\pi_{t_n \restriction k}}\right)\right)(N_{t_n})}.$$

The diagonal composition of the π_t is the desired $\wedge$-emb. $\square$

Thm (Basis for ${}^{\uparrow}\mathbb{N}_*^{\mathbb{N}} \setminus {}^{\uparrow}\mathbb{N}^{\mathbb{N}}$) X metric, $\phi : {}^{\uparrow}\mathbb{N}_*^{\mathbb{N}} \setminus {}^{\uparrow}\mathbb{N}^{\mathbb{N}} \longrightarrow X$
There exists a $\wedge$-embedding $\pi : {}^{\uparrow}\mathbb{N}^{<\mathbb{N}} \longrightarrow {}^{\uparrow}\mathbb{N}^{<\mathbb{N}}$
ST $\phi \circ \pi$ is either constant, or an injection in a discrete set, or a topological embedding.

Pf: We repeatedly precompose ϕ using the previous propositions. Each time we replace ϕ by its precomp. for notational simplicity.
By Cor. 8 we can suppose that ϕ is injective.
By Prop 9 we can suppose that $\operatorname{diam}(\phi(N_t))_t \longrightarrow 0$
By Prop 12, if $(\phi(t \frown (\infty)))_t$ converges to x, by injectivity there is a $\wedge$-emb. ST $\phi \circ \pi$ avoids x, so it is an injection in a discrete set.
So we can suppose that

$\forall m < n \quad \phi(t_m{}^\frown(\infty)) \notin \phi(N_{t_n})$.

So if $y_n \longrightarrow y$ in im ϕ, it means that there is a seq. $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^{\mathbb{N}}$ ST $\phi(t_{k_n}{}^\frown(\infty)) \longrightarrow \phi(t_k{}^\frown(\infty))$ for some $k \in \mathbb{N}$. So for no $i \in \mathbb{N}$ is there a subsequence of $(k_n)_{\mathbb{N}}$ contained in $N_{t_n{}^\frown(i)}$, in other words $t_{k_n}{}^\frown(\infty) \longrightarrow t_n{}^\frown(\infty)$.

So finally apply Prop 10, to get in the first case a topological embedding, and in the second an injection in a discrete set. $\square$

Let $f: \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}$ be an injection.

$\boxed{\text{\underline{Cor}: } \{ c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}},\ f,\ id_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}} \} \text{ is a } \leq\text{-basis for \underline{all} functions on } \mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}.}$

A basis for non-Baire class one function.

A function is Baire class one if it is Σ^0_2-measurable.

Given X, Y spaces, denote $X \oplus Y$ the space $\{0\} \times X \cup 1 \times \{Y\}$ considered as a subspace of $2 \times (X \cup Y)$.

Given two functions $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$ w/ $X_0 \cap X_1 = \emptyset$, the disjoint union is $f_0 \dot{\cup} f_1 : X_0 \cup X_1 \to Y_0 \oplus Y_1$

$$x \in X_i \longmapsto (i, f_i(x)).$$

Thm (Carroy-Miller) Assume that X and Y are analytic metric spaces, and $f: X \to Y$ is Borel.
Then there is a function ϕ_0 in $\{c_{\mathbb{N}^{\mathbb{N}}}, id_{\mathbb{N}^{\mathbb{N}}}\}$ and ϕ_1 in $\{c_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}, f, id_{\mathbb{N}^{\mathbb{N}}_* \setminus \mathbb{N}^{\mathbb{N}}}\}$ such that $\phi_0 \dot{\cup} \phi_1$ embeds in f.

Pf: There exists $U \subseteq Y$ open ST $f^{-1}(U)$ is not Σ^0_2.
Use Hurewicz dichotomy to get a continuous reduction $\tau : \mathbb{N}^{\mathbb{N}}_* \to X$ from $\mathbb{N}^{\mathbb{N}}$ to $f^{-1}(U)$, and Prop 5 to find a $\cap$-emb. π_0 ST $\tau \circ \pi_0 \restriction \mathbb{N}^{\mathbb{N}}$ is continuous.
Take $b \in \mathbb{N}^{\mathbb{N}}$, $V \ni f \circ \tau \circ \pi_0(b)$ whose closure is $\subseteq U$, and $n \in \mathbb{N}$ ST $f \circ \tau \circ \pi_0(N_{b \restriction n}) \subseteq V$, and set

$\pi_1 : s \mapsto b^\frown u^\frown s'$.

Use our two main theorems to stabilize the behaviour of $f \circ \Upsilon \circ \pi_0 \circ \pi_1$ restricted to N^N and its complement through π_2 and π_3 respectively. $\Upsilon \circ \pi_0 \circ \pi_1 \circ \pi_2 \circ \pi_3$ is the desired embedding. $\square$