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EMBEDDABILITY ON FUNCTIONS: ORDER AND CHAOS

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Abstract. We study the quasi-order of topological embeddability on definable functions between Polish 0-dimensional spaces.

We consider the descriptive complexity of this quasi-order restricted to the space of continuous functions. Our main result is the following dichotomy: the embeddability quasi-order restricted to continuous functions from a given compact space to another is either an analytic complete quasi-order or a well-quasi-order.

We also investigate the existence of maximal elements with respect to embeddability in a given Baire class. We prove that no Baire class admits a maximal element, except for the class of continuous functions which admits a maximum element.

Contents

1. Introduction

We study the quasi-order of (topological) embeddability on definable functions between Polish spaces. A function f is said to embed into a function g, in symbols $f \sqsubseteq g$, if there exist topological embeddings $\sigma : \operatorname{dom} f \to \operatorname{dom} g$ and $\tau : \operatorname{im} f \to \operatorname{im} g$ such that $g \circ \sigma = \tau \circ f$. Since the identity is an embedding and embeddings compose, embeddability is a reflexive and transitive relation on functions, namely, a quasi-order.

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This way of comparing functions was first used by Solecki [Sol98], who proved the existence of a non-σ-continuous Baire class 1 function which is minimal for embeddability. Several basis results of this sort were later obtained [PS12, CM17b], therefore confirming the relevance of this quasi-order for the study of definable functions.

In the first part of this article, we focus specifically on the properties of the quasi-order of embeddability when restricted to the space $C(X, Y)$ of continuous functions between two Polish 0-dimensional spaces X and Y. We endow these spaces of functions with the topology of compact convergence, which coincides in this case with the compact-open topology. While this turns $C(X, Y)$ into a Polish space if and only if X is locally compact, this choice is briefly discussed in Section 3.2.

Our first result shows that the embeddability on $C(X, Y)$ is quite complicated in most cases, in particular when both X and Y are uncountable. More precisely, let $\mathbb{G}$ denote the Polish space of graphs on $\mathbb{N}$ and consider the quasi-order of injective homomorphisms given by $G \preccurlyeq H$ if and only if there exists an injective homomorphism from G to H. Louveau and Rosendal [LR05, Theorem 3.5] showed that the homomorphism quasi-order on countable graphs is analytic complete, and their proof also shows that $(\mathbb{G}, \preccurlyeq)$ is an analytic complete quasi-order. We prove the following as Theorem 4.13.

Theorem 1.1. *Let X and Y be Polish 0-dimensional spaces. If X has infinitely many nonisolated points and Y is not discrete, then there exists a continuous map $\mathbb{G} \to C(X, Y)$ which reduces $\preccurlyeq$ to $\sqsubseteq$.*

Next we study the case where $C(X, Y)$ does not fall into the scope of the above theorem. Recall that a quasi-order is a well-quasi-order if it is well founded and has no infinite antichain. Of course an analytic complete quasi-order does not reduce to a well-quasi-order. In fact, while analytic complete quasi-orders are quite wild, the well-quasi-orders are very tame and provide a robust notion of hierarchy. In Section 5, we prove the following result as Theorems 5.7 and 5.8.

Theorem 1.2. *Let X and Y be Polish 0-dimensional spaces. If either X is locally compact with finitely many nonisolated points or Y is discrete, then embeddability is a well-quasi-order on $C(X, Y)$.*

Our methods unfortunately do not allow us to drop the assumption that X is locally compact in Theorem 1.2 (see Section 7).

However, combining our two theorems, we obtain a dichotomy for the spaces of functions $C(X, Y)$ with X locally compact. In particular, when X is compact, then $C(X, Y)$ is Polish and the topology coincides with that of the uniform convergence. We show that in this case embeddability is an analytic quasi-order (Theorem 3.2). We therefore obtain the following dichotomy.

Corollary 1.3. *Let X and Y be Polish 0-dimensional with X compact. Then exactly one of the following holds: either the embeddability on $C(X, Y)$ is analytic complete or it is a well-quasi-order.*

In Section 6, we move our attention to Borel functions on some Polish space and investigate the existence of maximal elements in Baire classes. Using the universality property of the Hilbert cube, we notice that the projection $p : [0, 1]^\omega \times [0, 1]^\omega \to$

$[0,1]^\omega$ is a maximum with respect to embeddability on continuous functions between Polish spaces. We then proceed to show that the situation is completely different for the other Baire classes (Theorem 6.10).

Theorem 1.4. *Let X, Y be Polish spaces such that X is uncountable and 0-dimensional and $|Y| \geq 2$. For every countable ordinal $\xi \geq 1$, there is no maximal element for embeddability among the Baire class ξ functions from X to Y.*

2. Preliminaries

2.1. **Spaces.** Unless otherwise specified, all topological spaces are assumed to be Polish 0-dimensional. We use variable symbols X, Y, Z for spaces. Our general reference for the descriptive set theory of these spaces is [Kec12]. For clarity, we write $(x_n)_n$ to denote a sequence and omit the index set, which is assumed to be $\mathbb{N}$. We write $x_n \to x$ in X to say that the sequence $(x_n)_n$ converges to x in a space X and we just write $x_n \to x$ when the space is clear from the context. Since all of our spaces are metrizable, a function $\sigma : X \to Y$ is a (topological) embedding if and only if for every $x \in X$ and every sequence $(x_n)_n$ in X, the following equivalence holds:

$$x_n \to x \text{ in } X \text{ if and only if } \sigma(x_n) \to \sigma(x) \text{ in } Y.$$

The notation $X = \bigsqcup_{i\in\mathbb{N}} X_i$ stands for the topological sum of a sequence $(X_i)_{i\in\mathbb{N}}$ of spaces, and we take $\bigcup_{i\in\mathbb{N}} \{i\} \times X_i$ to be the underlying set. This operation is functorial, and if we have continuous functions $f_i : X_i \to Y$ for all $i \in \mathbb{N}$, then $\bigsqcup_{i\in\mathbb{N}} f_i$ denotes the continuous map $\bigsqcup_i X_i \to Y$ given by $(i, x) \mapsto f_i(x)$.

If X is a locally compact metrizable space that is not compact, the Alexandrov compactification, or one-point compactification, of X is obtained by adding a new point ∞ to X, and consider the topology given by the sets $U \subseteq X$ open in X and the sets $V = \{\infty\} \cup X \setminus C$, where C is compact in X.

We view countable ordinals as topological spaces for the order topology. For example, ω is homeomorphic to $\mathbb{N}$ with the discrete topology, and $\omega + 1$ is homeomorphic to the one-point compactification of $\mathbb{N}$.

The ordinal ω^2 is homeomorphic to the sum of countably many copies of $\omega + 1$. Finally, the ordinal $\omega^2 + 1$ is homeomorphic to the one-point compactification of ω^2. Throughout the article, we freely identify ω^2 with $\bigsqcup_{n\in\mathbb{N}} \omega + 1$, and $\omega^2 + 1$ with the Alexandrov compactification of $\bigsqcup_{n\in\mathbb{N}} \omega + 1$.

We mention here for later use an easy lemma which allows us to glue together the embeddings.[1]

Lemma 2.1. *Let X be a Polish space. Suppose that F_0 and F_1 are two closed sets of X with $X = F_0 \cup F_1$ and $F_0 \cap F_1 = \{z\}$ for some $z \in X$. If $\sigma_i : F_i \to F_i$, $i = 0, 1$, are two embeddings such that $\sigma_0(z) = \sigma_1(z)$, then $\sigma = \sigma_0 \cup \sigma_1 : X \to Y$ is an embedding.*

Proof. Let $(x_n)_n$ be any sequence in X, and let $x \in X$. Let us show that $x_n \to x$ iff $\sigma(x_n) \to \sigma(x)$. In a case in which $(x_n)_n$ is eventually in F_0 or F_1, this follows from the fact that σ_i is an embedding for $i = 0, 1$. Otherwise, we can partition $(x_n)_n$ into two sequences, $(x^0_n)_n \subseteq F_0$ and $(x^1_n)_n \subseteq F_1$. Then $x_n \to x$ iff $x^i_n \to x$ for $i = 0, 1$. So in particular we must have $x = z$. Hence this is in turn equivalent to $\sigma_i(x^i_n) \to \sigma(z)$ for $i = 0, 1$, which is finally equivalent to $\sigma(x_n) \to \sigma(z)$. □

[1]Lemma 2.1 really says that the wedge sum of two embeddings between pointed metrizable spaces is again an embedding.

2.2. **Quasi-orders.** A *quasi-order* is a transitive reflexive relation $\leq_Q$ on some set Q. As is customary, we henceforth make an abuse of terminology and refer to the pair $(Q, \leq_Q)$ simply as Q when there is no danger of confusion.

We say that a quasi-order Q is *well founded* if every nonempty subset of Q admits a (not necessarily unique) $\leq_Q$-minimal element. A *well-quasi-order*, or WQO, is a well-founded quasi-order that has no infinite *antichain*, that is, no infinite subset of pairwise incomparable elements.

For clarity, we restrict our use of the terms "embed" and "embedding" to topological spaces or functions. Instead of referring to embeddings when dealing with quasi-orders, we talk about reductions. Given two quasi-orders P and Q, respectively, a *reduction* from $(P, \leq_P)$ to $(Q, \leq_Q)$ is a map $\varphi : P \to Q$ such that $p \leq_P p' \leftrightarrow \varphi(p) \leq_Q \varphi(p')$ for all $p, p' \in P$. We say that P *reduces* to Q if there exists a reduction from P to Q. When P and Q are topological spaces, we say that $(P, \leq_P)$ *continuously* (resp., *Borel*) *reduces* to $(Q, \leq_Q)$ when a continuous (resp., Borel) reduction exists.

A subset A of a Polish space X is analytic (or $\mathbf{\Sigma}^1_1$) if it is the projection of a closed subset of the product $\omega^\omega \times X$ of X with the Baire space. A quasi-order $\leq$ on some Polish space X is *analytic* if it is analytic as a subset of $X \times X$. It is *analytic complete* if it is analytic and moreover every analytic quasi-order on some Polish space Borel reduces to it.

Notice that since the quasi-order of equality on $\mathbb{N}$ does not reduce to any WQO, a WQO is never analytic complete.

We make use of the following example of analytic complete quasi-order in Section 4. It is defined on (simple) graphs with vertex set $\mathbb{N}$. These can be viewed as pairs $G = (\mathbb{N}, E)$, where the edge relation E is a subset of $[\mathbb{N}]^2$, the set of all pairs of natural numbers. Identifying any graph $G = (\mathbb{N}, E)$ with its edge relation E, we view G as an element of the Polish space $2^{[\mathbb{N}]^2}$. This makes the set of graphs on $\mathbb{N}$ a Polish space, and we denote it by $\mathbb{G}$.

An *injective homomorphism* from $G_0 = (\mathbb{N}, E_0)$ to $G_1 = (\mathbb{N}, E_1)$ is an injective map $h : \mathbb{N} \to \mathbb{N}$ such that $\{m, n\} \in E_0$ implies that $\{h(m), h(n)\} \in E_1$ for all pairs $\{m, n\}$. We write $G_0 \preccurlyeq G_1$ when there exists an injective homomorphism from G_0 to G_1. The proof of [LR05, Theorem 3.5] also gives the following result on which Section 4 relies.

Theorem 2.2 (Louveau–Rosendal)**.** *The quasi-order* $(\mathbb{G}, \preccurlyeq)$ *is analytic complete.*

2.3. **Some Cantor–Bendixson analysis.** We make a simple yet quite essential use of the Cantor–Bendixson analysis throughout the article. We briefly recall here what we need. We say that a point x in a space X is *isolated* if $\{x\}$ is open in X; otherwise, we say that x is a *limit*. We let $\mathrm{Is}(X)$ denote the set of isolated points of the space X.

The *Cantor–Bendixson sequence of derivatives* of a space X is defined inductively by

- $\mathrm{CB}_0(X) = X$,
- $\mathrm{CB}_{\alpha+1}(X) = \mathrm{CB}_\alpha(X) \setminus \mathrm{Is}(\mathrm{CB}_\alpha(X))$,
- $\mathrm{CB}_\alpha(X) = \bigcap_{\beta<\alpha} \mathrm{CB}_\beta(X)$ for cases in which α is a limit.

The *Cantor–Bendixson rank* of a space X is the smallest ordinal α such that $\mathrm{CB}_\alpha(X) = \mathrm{CB}_{\alpha+1}(X)$, and we denote it by $\mathrm{CB}(X)$. Since we are dealing only with second countable spaces, this rank will always be countable. Spaces of rank 0 are

either empty or perfect. The case of countable spaces is of particular interest for us; in that case, the sequence of Cantor–Bendixson derivatives is eventually empty, and the last nonempty derivative, if there is one, is always discrete.

The following results are the folklore.

Fact 2.3. We say that a space X is simple if $\mathrm{CB}(X) = \alpha + 1$ for some countable ordinal α, and $\mathrm{CB}_\alpha(X)$ is a singleton.

- If $\sigma : X \to Y$ is an embedding between countable spaces, then $\sigma(\mathrm{CB}_\alpha(X)) \subseteq \mathrm{CB}_\alpha(Y)$ for all countable α.
- Every countable Polish space is a topological sum of countably many simple spaces.
- If X is compact and countable, then $\mathrm{CB}(X) = \alpha + 1$ for some countable α, the cardinality of $\mathrm{CB}_\alpha(X)$ is finite, and X is the topological sum of $|\mathrm{CB}_\alpha(X)|$ simple spaces of rank $\alpha + 1$.

For the proof, in a slightly more general context, see, for instance, [Car13, Proposition 2.2 and Lemma 2.4].

The following classification of Polish spaces with exactly one limit point[2] is used in Section 4. Here we denote by $[\omega]^2_+$ the set $[\omega]^2 \cup \{\emptyset\}$ viewed as a subspace of the Cantor space.

Proposition 2.4. *Up to homeomorphism, there are only three Polish spaces with a single limit point, namely,*

$$\omega + 1,\ \omega \sqcup (\omega + 1) \text{ and } [\omega]^2_+.$$

Notice that $\omega + 1$ is compact and $\omega \sqcup (\omega + 1)$ is locally compact, while $[\omega]^2_+$ is not locally compact.

Proof. Let X be Polish with a single limit point x_∞, and fix a compatible metric for X. Take a sequence $(U_n)_{n\in\mathbb{N}}$ of open balls centered at x_∞ with vanishing diameter. Notice that for all n, $X \setminus U_n$ is a countable set of isolated points of X; hence it is open. Therefore, each U_n is actually clopen in X, and so are the sets $V_n = U_n \setminus U_{n+1}$. We distinguish three cases.

Suppose first that V_n is finite for all $n \in \mathbb{N}$ and thus that X is compact. Then any bijective map $h : X \to \omega + 1$ which maps x_∞ to ω is a homeomorphism.

Next assume that there is an $N \in \mathbb{N}$ such that V_n is finite for all $n \geq N$, while $X \setminus U_N$ is infinite. Now $X \setminus U_N$ is clopen in X, and as an infinite discrete space it is homeomorphic to ω. Since U_N falls under the first case, X is homeomorphic to $\omega \sqcup (\omega + 1)$.

Otherwise, by possibly going to a subsequence of neighborhoods $(U_n)_{n\in\mathbb{N}}$, we can assume that V_n is infinite for all n. Fix an enumeration $(x_{m,n})_{n\in\mathbb{N}}$ of V_m for all m, and define $h : X \to [\omega]^2_+$ by $\tau(x_{m,n}) = \{m, m + n\}$ and $h(x_\infty) = \emptyset$. One easily checks that h is a homeomorphism, as desired. □

3. Embedding functions

3.1. **Let us embed them.** The following ordering on functions is the main object of study in this article.

[2]Or equivalently, separable metrizable spaces with exactly one limit point.

Definition 3.1. Let $f : X \to Y$ and $g : X' \to Y'$ be two functions between topological spaces. We say that f embeds in g, in symbols $f \sqsubseteq g$, if there exist embeddings $\sigma : X \to X'$ and $\tau : \operatorname{im} f \to \operatorname{im} g$ such that $\tau \circ f = g \circ \sigma$. In this case, we also say that (σ, τ) embeds f in g.

We start by making a simple but important observation on this definition. If (σ, τ) embeds f into g, then the embedding $\tau : \operatorname{im} f \to \operatorname{im} g$ is entirely determined by σ, f, and g through the equation $\tau(f(x)) = g(\sigma(x))$ for all $x \in X$. In particular, if (σ, τ) and (σ, τ') both embed f into g, then $\tau = \tau'$. With this in mind, we sometimes just say that σ embeds f into g if σ is an embedding and there exists a (unique) embedding $\tau : \operatorname{im} f \to \operatorname{im} g$ such that $\tau \circ f = g \circ \sigma$.

Observe that in order for an embedding $\sigma : X \to X'$ to embed f into g, it is necessary that σ reduces the equivalence relation induced by f to that induced by g, namely, that

$$\forall x_0, x_1 \in X \qquad \big(f(x_0) = f(x_1) \leftrightarrow g(\sigma(x_0)) = g(\sigma(x_1))\big),$$

in which case, the assignment $f(x) \mapsto g(\sigma(x))$ actually defines an injective function $\tau : \operatorname{im} f \to \operatorname{im} g$. Hence for σ to embed f into g, it is necessary and sufficient that moreover this τ is an embedding.

Now given Polish spaces X and Y, we endow the space $C(X, Y)$ of continuous functions from X to Y with the topology of compact convergence. This topology coincides in this case with the compact-open topology (see [Mun00, Theorem 46.8, p. 285]), which is defined by taking as a subbase sets of the form

$$S_{X,Y}(C, U) = \{f : X \to Y \mid f(C) \subseteq U\}$$

for some compact subset C of X and some open subset U of Y. We recall that if X is Hausdorff and $\mathcal{S}$ is a subbase for the topology of Y, then the sets of the form $S_{X,Y}(C, U)$, where U belongs to $\mathcal{S}$, already form a subbase for the compact-open topology of $C(X, Y)$ [Jac52, Lemma 2.1].

When X is locally compact and Y is Polish, then $C(X, Y)$ is Polish (see [Kur68, Theorem 1, p. 93, and Theorem 3, p. 94]), while in the nonlocally compact case this topology is not even metrizable. We address the existence of a natural topology on these spaces at the end of this section.

When moreover X is compact, then the topology on $C(X, Y)$ coincides with the uniform convergence topology induced by the uniform metric

$$d_u(f, g) = \sup_{x \in X} d_Y(f(x), g(x)),$$

where d_Y is some compatible metric for Y. In this case, $C(X, Y)$ is Polish [Kec12, (4.19)], and we show that the quasi-order of embeddability is analytic. Notice that our general assumption that the spaces are 0-dimensional is actually unnecessary for this result.

Theorem 3.2. *Let X and Y be Polish, with X compact. Then embeddability is an analytic quasi-order on $C(X, Y)$.*

Proof. We show that the relation E on $C(X, X) \times C(X, Y)^2$ given by

$$(\sigma, f, g) \in E \iff \sigma \text{ embeds } f \text{ into } g$$

is Borel. Since $f \sqsubseteq g$ iff $\exists \sigma \in C(X, X)\ (\sigma, f, g) \in E$, this implies that the relation $f \sqsubseteq g$ is analytic, as desired.

Fix a countable dense subset $D = \{d_n : n \in \omega\}$ of X and compatible complete metrics d_X and d_Y on X and Y, respectively.

First, we observe that for every $x, x' \in X$ and $k > 0$ the set $D^k_{x,x'} = \{f \in C(X,Y) : d_Y(f(x), f(x')) \leq \frac{1}{k}\}$ is closed in $C(X,Y)$. To see this, let $f \in C(X,Y)$, with $d_Y(f(x), f(x')) \geq \frac{1}{k} + \epsilon$ for some $\epsilon > 0$. Then for every $g \in C(X,Y)$ with $d_u(f,g) < \frac{\epsilon}{2}$, we have $d_Y(g(x), g(x')) \geq d_Y(f(x), f(x')) - 2 \cdot d_u(f,g) > \frac{1}{k}$. Hence the complement of $D^k_{x,x'}$ is open, as desired.

Next we prove that for all $\sigma \in C(X,X)$, σ is an embedding if and only if

$$\forall k > 0,\ \exists l > 0,\ \forall m, n\ \Big(d_Y(\sigma(d_n), \sigma(d_m)) \leq \frac{1}{l} \text{ implies that } d_X(d_n, d_m) \leq \frac{1}{k}\Big),$$

which shows that the set of embeddings is Borel in $C(X,X)$. Suppose that $\sigma \in C(X,X)$ is an embedding and therefore admits a continuous inverse $\sigma^{-1} : \operatorname{im}\sigma \to X$. Since X is compact, so is $\operatorname{im}\sigma$, and hence σ^{-1} is uniformly continuous. This shows the forward implication. Conversely, assume that $\sigma \in C(X,X)$ satisfies the above formula. By compactness of X, it suffices to show that σ is injective. To this end, assume that $\sigma(x) = \sigma(x')$ and pick two sequences in D with $b_n \to x$ and $c_m \to x'$. Since both $(\sigma(b_n))_n$ and $(\sigma(c_m))_m$ converge to $\sigma(x)$, it follows that the sequence $(\sigma(b_0), \sigma(c_0), \sigma(b_1), \sigma(c_1), \ldots)$ is Cauchy. As σ satisfies the above formula, it follows that $(d_0, c_0, d_1, c_1, \ldots)$ is Cauchy too, which implies that $x = x'$.

Finally, we claim that $(\sigma, f, g) \in E$ if and only if σ is an embedding,

$$\forall k > 0,\ \exists l > 0,\ \forall n, m\ \Big(d_Y(f(d_n), f(d_m)) \leq \frac{1}{l} \to d_Y(g \circ \sigma(d_n), g \circ \sigma(d_m)) \leq \frac{1}{k}\Big)$$

and

$$\forall k > 0,\ \exists l > 0,\ \forall n, m\ \Big(d_Y(g \circ \sigma(d_n), g \circ \sigma(d_m)) \leq \frac{1}{l} \to d_Y(f(d_n), f(d_m)) \leq \frac{1}{k}\Big).$$

This shows that E is Borel. Suppose first that $(\sigma, f, g) \in E$. Then σ is an embedding and we have a unique embedding $\tau : \operatorname{im} f \to \operatorname{im} g \circ \sigma$ such that (σ, τ) embeds f into g. Since $\operatorname{im} f$ and $\operatorname{im} g \circ \sigma$ are compact, both τ and its inverse are uniformly continuous. Moreover as $\tau(f(x)) = g \circ \sigma(x)$ for all $x \in X$, it follows that the two above formulas are satisfied.

Conversely, suppose that $\sigma : X \to X$ is an embedding such that the two above formulas hold with respect to some f and g in $C(X,Y)$. Notice that in particular, we have $f(d_n) = f(d_m)$ iff $g \circ \sigma(d_n) = g \circ \sigma(d_m)$ for every $m, n \in \omega$. We therefore have a well defined bijective map $\tilde{\tau} : f(D) \to g \circ \sigma(D)$ given by $\tilde{\tau}(f(d)) = g \circ \sigma(d)$. Since $f(D)$ is dense in $\operatorname{im} f$ and $\tilde{\tau}$ is uniformly continuous, it extends to a continuous map $\tau : \operatorname{im} f \to \operatorname{im} g \circ \sigma$. Moreover as before, this continuous extension is injective, and so τ is an embedding. Finally, since $\tau \circ f$ and $g \circ \sigma$ are both continuous and coincide on D, they are equal on X. Therefore, τ is an embedding such that (σ, τ) embeds f in g, as desired. □

Remark 3.3. When X and Y are Polish and X is locally compact, the space $C(X,Y)$ is Polish. One might therefore be tempted to think that the embeddability relation is analytic for such spaces as well. However, if we assume only that X is locally compact in Theorem 3.2, then the conclusion does not hold. There exists a locally compact Polish space X for which embeddability is not analytic on $C(X,X)$ (see Section 7).

We end this subsection by collecting a few basic facts for later use. Observe that any embedding $j : Y \to Z$ induces a map $j^\circ : C(X,Y) \to C(X,Z)$ given by $j^\circ(f) = j \circ f$. It should come as no surprise that this map actually is a continuous reduction with respect to embeddability on functions.

Proposition 3.4. *For all spaces X, Y, Z, if Y embeds in Z, then $(C(X,Y), \sqsubseteq)$ continuously reduces to $(C(X,Z), \sqsubseteq)$.*

Proof. Let $j : Y \to Z$ be an embedding. Then the induced map $j^\circ : C(X,Y) \to C(X,Z)$ given by $j^\circ(f) = j \circ f$ is continuous by [MN06, Theorem 2.2.4.b, p. 18].

Now for any embedding $\sigma : X \to X$, we show that σ embeds f into g if and only if σ embeds $j^\circ(f)$ into $j^\circ(g)$. Since j is an embedding, we have

$$\forall x_0, x_1 \in X \qquad f(x_0) = f(x_1) \iff g(\sigma(x_0)) = g(\sigma(x_1))$$

if and only if

$$\forall x_0, x_1 \in X \qquad j^\circ(f)(x_0) = j^\circ(f)(x_1) \iff j^\circ(g)(\sigma(x_0)) = j^\circ(g)(\sigma(x_1)).$$

Therefore, an embedding $\sigma : X \to X$ induces a injective function $\tau : \operatorname{im} f \to \operatorname{im} g$ such that $\tau \circ f = g \circ \sigma$ if and only if it induces an injective function $\tau' : \operatorname{im} j^\circ(f) \to \operatorname{im} j^\circ(g)$ with $\tau' \circ j^\circ(f) = j^\circ(g) \circ \sigma$. Now notice that τ is an embedding just in case for all sequences $(x_n)_n$ in X and all $x \in X$, $f(x_n) \to f(x)$ iff $g(\sigma(x_n)) \to g(\sigma(x))$. Since j is an embedding, we see that τ is an embedding exactly when is τ' also. Therefore, $f \sqsubseteq g$ iff $j^\circ(f) \sqsubseteq j^\circ(g)$, so j° is a reduction, as desired. □

In Section 4, we will make use of functions which are almost embeddings while they lack injectivity. Whenever a function $\rho : X \to Y$ is not injective, we can pick distinct points x and x' in X with $\rho(x) = \rho(x')$, and therefore the sequence $(x, x', x, x', \ldots)$ does not converge while $(\rho(x), \rho(x'), \rho(x), \rho(x'), \ldots)$ is actually constant. With this very simple observation in mind, we make the following definition.

Definition 3.5. Let $\rho : X \to X'$ be a function. We say that a sequence $(x_n)_n$ in X is *ρ-trivial* if the corresponding sequence $\big(\rho(x_n)\big)_n$ of images by ρ is ultimately constant. We say that ρ is a *pseudo-embedding* if ρ is continuous and for all non-ρ-trivial sequences $(x_n)_n$ and all $x \in X$ we have

$$x_n \to x \quad \text{if and only if} \quad \rho(x_n) \to \rho(x).$$

Notice that an injective pseudo-embedding is just an embedding. One reason for considering this weakening of the notion of embedding is that the relation of embeddability simplifies a little when restricted to such functions.

Proposition 3.6. *Let $f : X \to Y$ and $g : X' \to Y'$ be two pseudo-embeddings. Then $f \sqsubseteq g$ if and only if there exists an embedding $\sigma : X \to X'$ such that $f(x) = f(x')$ iff $g(\sigma(x)) = g(\sigma(x'))$ for all $x, x' \in X$.*

Proof. Clearly this is necessary, so let us prove that this actually is sufficient. Let $\sigma : X \to X$ be an embedding such that $f(x) = f(x')$ iff $g(\sigma(x)) = g(\sigma(x'))$ for all $x, x' \in X$. This ensures that the assignment $f(x) \mapsto g(\sigma(x))$ defines an injective map $\tau : \operatorname{im} f \to \operatorname{im} g$. To see that this τ is an embedding, take some sequence $(x_n)_n$ in X and let $x \in X$, with the aim of showing that

$$f(x_n) \to f(x) \text{ in } Y \iff g(\sigma(x_n)) \to g(\sigma(x)) \text{ in } X.$$

Notice that $(x_n)_n$ is f-trivial if and only if $(\sigma(x_n))_n$ is g-trivial. If $(x_n)_n$ is f-trivial and thus $(\sigma(x_n))_n$ is g-trivial too, then the sequences on both sides are eventually

constant and we are done. So suppose that $(x_n)_n$ is not f-trivial and thus $(\sigma(x_n))_n$ is not g-trivial either. Since f and g are pseudo-embeddings, we see that each side of the equivalence hold if and only if $x_n \to x$, which finishes the proof. □

If $\rho : X \to Y$ is a pseudo-embedding and $\tau : Y \to Y$ is an embedding, then there is an easy sufficient (and in fact necessary) condition for the existence of an embedding $\sigma : X \to X$ such that (σ, τ) is a witness for $\rho \sqsubseteq \rho$.

Proposition 3.7. *Let $\rho : X \to Y$ be a pseudo-embedding. If $\tau : Y \to Y$ is an embedding such that $\rho^{-1}(y)$ embeds in $\rho^{-1}(\tau(y))$ for every $y \in Y$, then there exists an embedding $\sigma : X \to X$ such that $\rho \circ \sigma = \tau \circ \rho$.*

Proof. Pick for every $y \in \operatorname{im} \rho$ an embedding $\sigma_y : \rho^{-1}(y) \to \rho^{-1}(\tau(y))$. We show that $\sigma = \bigcup_{y \in Y} \sigma_y$ is an embedding such that (σ, τ) embeds ρ in itself. Clearly we have $\rho \circ \sigma = \tau \circ \rho$. To see that σ is an embedding, let $(x_n)_n$ be a sequence in X and $x \in X$. We claim that $x_n \to x$ iff $\sigma(x_n) \to \sigma(x)$. Observe that $(x_n)_n$ is ρ-trivial exactly when $(\sigma(x_n))_n$ is. If $(x_n)_n$ is ρ-trivial with $\big(\rho(x_n)\big)_n$ eventually equal to y, then the claim follows from the fact that σ_y is an embedding. Otherwise, since ρ is a pseudo-embedding $x_n \to x$ iff $\rho(x_n) \to \rho(x)$, as τ is an embedding, this is equivalent to $\tau(\rho(x_n)) \to \tau(\rho(x))$. This is in turn equivalent to $\rho \circ \sigma(x_n) \to \rho \circ \sigma(x)$; hence using that ρ is a pseudo-embedding once again, we get this to finally be equivalent to $\sigma(x_n) \to \sigma(x)$. □

3.2. **A limitation result on possible topologies for continuous functions.** A natural question arises when we consider spaces of the form $C(X, Y)$: does there exist a nice topology on these spaces? More precisely, does there exist a standard Borel structure on $C(X, Y)$ so that the *evaluation map* $C(X, Y) \times X \to Y$ defined by $(f, x) \mapsto f(x)$ is Borel?

We first observe that the answer is affirmative when X is σ-compact. Let $X = \bigcup_{n \in \omega} K_n$ with compact sets K_n, and identify $C(X, Y)$ with a subset of the Polish space $\prod_{n \in \omega} C(K_n, Y)$ via the injection $f \mapsto (f|_{K_n})_n$. It is enough to notice the following fact.

Proposition 3.8. *The space $C(X, Y)$ is Borel in $\prod_{n \in \omega} C(K_n, Y)$.*

Proof. Clearly $C(X, Y)$ is a coanalytic subset of $\prod_{n \in \omega} C(K_n, Y)$, and we show that it is also analytic, and hence Borel. Fix metrics d_X and d_Y on X and Y, and countable dense subsets $D_i \subseteq K_i$. We claim that for every $(f_i)_i \in \prod_{n \in \omega} C(K_n, Y)$, $\bigcup_{i \in \mathbb{N}} f_i$ is the graph of a continuous function if and only if for every $n > 0$ there is a sequence $(\alpha_i)_i$ with $\alpha_i > 0$ such that for every $i, j \in \mathbb{N}$ if $d \in D_i$, $d' \in D_j$, and $d_X(d, d') < \frac{1}{\alpha_i}$, then $d_Y(f_i(d), f_j(d')) < \frac{1}{n}$. This definition is analytic, as desired.

Suppose that $(f_i)_i$ satisfies the above analytic definition. We claim that for all i, for all $x \in K_i$, and for all $N > 0$ there exists a $\delta > 0$ such that $x' \in K_{i'}$, and $d_X(x, x') \leq \delta$ implies that $d_Y(f_i(x), f_{i'}(x')) < \frac{1}{N}$. This implies that $\bigcup_i f_i$ is the graph of a continuous function. Toward a contradiction, suppose that for some i there are an $x \in K_i$ and an $N > 0$ witnessing the failure of the claim. Using our assumption for $n = 3N$, we get a sequence $(\alpha_i)_i$ witnessing that $(f_i)_i$ satisfies our analytic definition. Now by our choice of i, x, and N, there exist i' and $x' \in K_{i'}$ with $d_X(x, x') < \frac{1}{3\alpha_i}$ and $d_Y(f_i(x), f_{i'}(x')) \geq \frac{1}{N}$. By continuity of f_i on K_i and density of D_i, there exists a $d \in D_i$ with $d_X(x, d) < \frac{1}{3 \cdot \alpha_i}$ and $d_Y(f_i(x), f_i(d)) < \frac{1}{3N}$. Similarly, there is a $d' \in D_{i'}$ with $d_X(x', d') < \frac{1}{3 \cdot \alpha_i}$ and $d_Y(f_{i'}(x'), f_{i'}(d')) < \frac{1}{3N}$.

Hence it follows that $d_X(d, d') < 1/\alpha_i$ but $d_Y(f_i(d), f_{i'}(d')) \geq \frac{1}{3N}$, which is a contradiction.

Conversely, assume that $f : X \to Y$ is continuous, and fix $n > 0$. By continuity of f, for each $x \in X$ there exists a δ_x such that $\operatorname{diam}(f(B(x, \delta_x))) < \frac{1}{n}$. Fix $i \in \mathbb{N}$, and notice that $K_i \subseteq \bigcup_{x \in K_i} B(x, \frac{1}{2} \cdot \delta_x)$. By compactness, K_i is already covered by finitely many open balls centered at some $(x_k)_{k<m}$, and we take $\alpha_i > 0$ such that $\frac{1}{\alpha_i} < \frac{1}{2} \cdot \min\{\delta_{x_k} \mid k < m\}$. Then if $x \in K_i$, then for some $k < m$ we have $x \in B(x_k, \frac{1}{2} \cdot \delta_{x_k})$, so in particular $B(x, \frac{1}{\alpha_i}) \subseteq B(x_k, \delta_{x_k})$, which implies that $\operatorname{diam}(f(B(x, \frac{1}{\alpha_i}))) < \frac{1}{n}$. □

Therefore, when X is a σ-compact metric space and Y is Polish, there is a standard Borel structure on $C(X, Y)$ which makes the evaluation map Borel. We next show that this fails if X is not σ-compact and 0-dimensional. In particular, there is no standard Borel structure on $C(\omega^\omega, \omega^\omega)$ which makes the evaluation map Borel. As usual, if $B \subseteq X \times Y$, $x \in X$, and $y \in Y$, we denote the horizontal and vertical sections, that is, the sets $\{x' \in X : (x', y) \in B\}$ and $\{y' \in Y : (x, y') \in B\}$, by B^y and B_x, respectively.

Theorem 3.9. *Let X, Y be 0-dimensional Polish spaces with X non-σ-compact and $|Y| \geq 2$. Then there is no standard Borel structure on $C(X, Y)$ such that the map $C(X, Y) \times X \to Y$ defined by $(f, x) \mapsto f(x)$ is Borel.*

Proof. Suppose toward a contradiction that there is such a structure. First, we reduce the problem to $X = \omega^\omega$. By the isomorphism of standard Borel spaces, there exists a Borel bijection $b : \omega^\omega \to C(X, Y)$. As X is not σ-compact, it contains a closed subspace X' homeomorphic to ω^ω, so let $e : \omega^\omega \to X'$ be such a homeomorphism. We define a subset B of $\omega^\omega \times \omega^\omega \times Y$ by $(r, s, y) \in B \iff b(r)(e(s)) = y$. Then by our assumption, B is a Borel set and of course, for every $r \in \omega^\omega$, we have B_r being the graph of a continuous function from ω^ω to Y. Now as X' is a closed subset of X, every continuous function $X' \to Y$ extends to a continuous function $X \to Y$ (by [Kur66, Corollary 6c, p. 151], [Kec12, (4.17)]). Consequently, for every $f : \omega^\omega \to Y$ continuous, there exists an r such that B_r is the graph of f.

Therefore, it is enough to prove that there is no such Borel set $B \subseteq \omega^\omega \times \omega^\omega \times Y$. Fix a continuous onto map $t : Y \to 2$, and consider the following set:

$$B' = \{(r, s) \in (\omega^\omega)^2 : (\forall y)((r, s, y) \in B \implies t(y) = 0)\}.$$

We have the following:

- B' is Borel since $(r, s) \in B' \iff (\exists y)((r, s, y) \in B \implies t(y) = 0)$;
- for every r, the section B'_r is clopen;
- for every clopen subset C of ω^ω, there exists an r with $B'_r = C$.

Without loss of generality, we can suppose that for every clopen set C the set $\{r \in \omega^\omega : B'_r = C\}$ cannot be covered by the union of countably many compact sets: indeed, instead of B' we can consider the set $B'' \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ defined by $(r', r, s) \in B'' \iff (r, s) \in B'$. Then for each clopen C the set $\{(r', r) : B''_{(r,r')} = C\}$ contains a copy of ω^ω. Identifying $(\omega^\omega)^2$ with ω^ω via a standard homeomorphism gives the desired set.

Let us denote the usual topology on ω^ω by τ_u. Now, every section B'_x is clopen, and it follows from [Kec12, Theorem 28.7] that there exists a Polish topology $\tau_0 \supseteq \tau_u$ on ω^ω so that B' is open in $(\omega^\omega, \tau_0) \times (\omega^\omega, \tau_u)$. Applying this again for $(B')^c$, we

can get a 0-dimensional Polish topology $\tau \supseteq \tau_0$ on ω^ω such that B' is clopen in the product space $(\omega^\omega, \tau) \times (\omega^\omega, \tau_u)$. Note that if a subset of ω^ω is compact with respect to τ, then it is also compact with respect to τ_u. Hence, for every clopen C, the set $\{x : B'_x = C\}$ cannot be covered by countably many τ-compact sets. Now, since (ω^ω, τ) is 0-dimensional, there exists a σ-compact τ-open set S such that $(\omega^\omega \setminus S, \tau)$ is homeomorphic to (ω^ω, τ_u), so let us denote such a homeomorphism by h.

Define $(x, y) \in U \iff (h(x), y) \in B'$. Now, U is a clopen subset of $(\omega^\omega)^2$ (indeed, it is the preimage of a clopen set under the map (h, id)). For every C clopen, there exists an r with $U_r = C$: as S is σ-compact, for any C there is an r with $B_r = C$ such that $r \notin S$. This contradicts the fact that there is no universal clopen set. □

4. Chaos

Our starting point is a general and simple observation about the embeddability quasi-order on (not necessarily continuous or definable) functions between topological spaces.

Definition 4.1 (The function-to-graph map)**.** With every function $f : X \to Y$ between topological spaces we associate a graph $\mathcal{G}_f = (V_f, E_f)$, where $V_f = \{x \in X \mid x \text{ is a limit point}\}$ and

$$\{x, y\} \in E_f \iff x \neq y, \text{ and there exist injective sequences } (x_n)_n \text{ and } (y_n)_n \text{ in } X \text{ such that } x_n \to x,\ y_n \to y, \text{ and } f(x_n) = f(y_n) \text{ for all } n \in \mathbb{N}.$$

Notice that if $(x_n)_n$ and $(y_n)_n$ witness that $\{x, y\} \in E_f$, then so do $(x_{n_k})_k$ and $(y_{n_k})_k$ for any strictly increasing sequence $(n_k)_k$ of natural numbers.

The map $f \mapsto \mathcal{G}_f$ is actually a homomorphism from $\sqsubseteq$ to $\preccurlyeq$.

Lemma 4.2. *Let $f : X \to Y$ and $f' : X' \to Y'$ be two functions between topological spaces. Then $f \sqsubseteq f'$ implies that $\mathcal{G}_f \preccurlyeq \mathcal{G}_{f'}$.*

Proof. Let (σ, τ) embed f into f'. Let us denote by $\check{\sigma} : V_f \to V_{f'}$ the restriction of $\sigma : X \to X'$ to the limit points of X. This is well defined and injective since σ is an embedding, so in particular σ must send any limit point of X to some limit point of X'. To see that it is a homomorphism from $\mathcal{G}_f$ to $\mathcal{G}_{f'}$, suppose that $\{x, y\} \in E_f$. There exist injective sequences $(x_n)_n$ and $(y_n)_n$ such that $x_n \to x$, $y_n \to y$, and $f(x_n) = f(y_n)$ for all $n \in \mathbb{N}$. As σ is injective and continuous, it follows that $(\sigma(x_n))_n$ and $(\sigma(y_n))_n$ are injective sequences converging to $\sigma(x)$ and $\sigma(y)$, respectively. Moreover, $f'(\sigma(x_n)) = \tau(f(x_n)) = \tau(f(y_n)) = f'(\sigma(y_n))$ for every $n \in \mathbb{N}$, so $\{\sigma(x), \sigma(y)\} \in E_{f'}$, as desired. □

In this section, we prove that if Y is not discrete and X has infinitely many limit points, then $(\mathbb{G}, \preccurlyeq)$ continuously reduces to $(C(X, Y), \sqsubseteq)$. Notice that Y is not discrete if and only if $\omega + 1$ embeds in Y, so by Proposition 3.4 it is enough to show that the result holds for $C(X, \omega + 1)$ when X has infinitely many limit points.

4.1. **One special case, or maybe two.** We define a homomorphism $G \mapsto f^G$ from $(\mathbb{G}, \preccurlyeq)$ to $(C(\omega^2, \omega + 1), \sqsubseteq)$. We then show that G is isomorphic to $\mathcal{G}_{f^G}$ for every $G \in \mathbb{G}$. As $f \mapsto \mathcal{G}_f$ is a homomorphism by Lemma 4.2, it will follow that $G \mapsto f^G$ is actually a reduction, as desired.

Remember that we identify ω^2 with $\bigsqcup_{n\in\mathbb{N}} \omega+1$. The homomorphism is defined as follows: each limit point (m,ω) corresponds to the vertex m, and we pick for every other vertex n a sequence of isolated points converging to (m,ω). If $\{m,n\}$ is an edge, then the sequence converging to (m,ω) that we picked for n and the sequence converging to (n,ω) that we picked for m are mapped to the same converging sequence in $\omega+1$. Otherwise, they are mapped to disjoint sequences. More formally, we have the following definition.

Definition 4.3 (The graph-to-function map)**.** Fix bijections $\langle -\rangle_0 : \mathbb{N}^2 \to \mathbb{N}$, $\langle -\rangle_1 : [\mathbb{N}]^2 \to \mathbb{N}$ and $\langle -\rangle_2 : 2\times\mathbb{N}^2 \to \mathbb{N}$. For clarity, we write $\langle m,n\rangle_1$ instead of $\langle\{m,n\}\rangle_1$. Note that if $n \neq m$, then $\langle m,n\rangle_0 \neq \langle n,m\rangle_0$, while $\langle m,n\rangle_1 = \langle n,m\rangle_1$. For every vertex $m \in \mathbb{N}$, define $f^G_m : \omega+1 \to \omega+1$ by

$$\langle n,p\rangle_0 \longmapsto \begin{cases} \langle 0, \langle m,n\rangle_0, p\rangle_2 & \text{if } \{m,n\} \notin E^G, \\ \langle 1, \langle m,n\rangle_1, p\rangle_2 & \text{if } \{m,n\} \in E^G, \end{cases}$$
$$\omega \longmapsto \omega.$$

We define $f^G : \omega^2 \to \omega+1$ to be the sum $\bigsqcup_{n\in\mathbb{N}} f^G_n : \bigsqcup_{n\in\mathbb{N}} \omega+1 \to \omega+1$.

Clearly each f^G_n is continuous, so f^G is continuous.

Lemma 4.4. *For every $G, G' \in \mathbb{G}$, if $G \preccurlyeq G'$, then $f^G \sqsubseteq f^{G'}$.*

Proof. Suppose that $\sigma : \mathbb{N} \to \mathbb{N}$ is an injective homomorphism from G to G'. Define $\check{\sigma} : \omega^2 \to \omega^2$ by $\check{\sigma}(m,\omega) = (\sigma(m),\omega)$ for every $m \in \mathbb{N}$ and

$$\check{\sigma}(m,\langle n,p\rangle_0) = \begin{cases} (\sigma(m), \langle \sigma(n), 2p\rangle_0) & \text{if } m<n,\ \{m,n\}\notin E^G \\ & \text{and } \{\sigma(m),\sigma(n)\}\in E^{G'}, \\ (\sigma(m), \langle \sigma(n), 2p+1\rangle_0) & \text{if } n<m,\ \{m,n\}\notin E^G \\ & \text{and } \{\sigma(m),\sigma(n)\}\in E^{G'}, \\ (\sigma(m), \langle \sigma(n), p\rangle_0) & \text{otherwise} \end{cases}$$

for every $m,n,p \in \mathbb{N}$. Then define $\tau\omega+1 \longrightarrow \omega+1$ by $\tau(\omega)=\omega$, $\tau\langle 1,\langle m,n\rangle_1,p\rangle_2 = \langle 1,\langle\sigma(m),\sigma(n)\rangle_1,p\rangle_2$ and

$$\tau(\langle 0,\langle m,n\rangle_0,p\rangle_2) = \begin{cases} \langle 1,\langle\sigma(m),\sigma(n)\rangle_1, 2p\rangle_2 & \text{if } m<n,\ \{m,n\}\notin E^G \\ & \text{and } \{\sigma(m),\sigma(n)\}\in E^{G'}, \\ \langle 1,\langle\sigma(m),\sigma(n)\rangle_1, 2p+1\rangle_2 & \text{if } n<m,\ \{m,n\}\notin E^G \\ & \text{and } \{\sigma(m),\sigma(n)\}\in E^{G'}, \\ \langle 0,\langle\sigma(m),\sigma(n)\rangle_0, p\rangle_2 & \text{otherwise.} \end{cases}$$

One easily checks that $(\check{\sigma},\tau)$ embeds f^G into $f^{G'}$. □

Lemma 4.5. *For every graph $G = (\mathbb{N}, E^G)$ in $\mathbb{G}$, the map*

$$j : \mathbb{N} \longrightarrow V_{f^G},$$
$$n \longmapsto (n,\omega),$$

is an isomorphism of graphs from G to $\mathcal{G}_{f^G}$.

Proof. Notice that $j : \mathbb{N} \to V_{f^G}$ is bijective. First, take an edge $\{m,n\} \in E^G$. Consider the two sequences given by $x_p = (m,\langle n,p\rangle_0)$ and $y_p = (n,\langle m,p\rangle_0)$, $p \in \mathbb{N}$. These sequences are injective and converge, respectively, to $j(m)$ and $j(n)$.

Moreover, by the definition of f^G, we have $f^G(x_p) = f^G(y_p)$ for every $p \in \mathbb{N}$; hence $\{j(m), j(n)\} \in E_{f^G}$.

Conversely, suppose that $(x_p)_p$ and $(y_p)_p$ are injective sequences in $\bigsqcup_n \omega + 1$ witnessing that $\{j(m), j(n)\} \in E_{f^G}$. Since $x_p \to (m, \omega)$ and $y_p \to (n, \omega)$, we can assume by simultaneously passing to subsequences that $(x_p)_p \subseteq \{m\} \times \omega$ and $(y_p)_p \subseteq \{n\} \times \omega$. In particular, $x_0 = (m, \langle k, p\rangle_0)$ and $y_0 = (n, \langle k', p'\rangle_0)$ for some $k, k', p, p' \in \mathbb{N}$. But $f^G(x_0) = f^G(y_0)$, so by the definition of f^G, this is possible only if $p = p'$, $k' = m$, and $k = n$. Consequently, $\{m, n\} \in E^G$. □

While the reduction from $(\mathbb{G}, \preccurlyeq)$ to $C(\omega^2, \omega + 1)$ is simple and illuminating (at least we hope so), we will actually need a reduction to $C(\omega^2 + 1, \omega + 1)$ in order to derive the general case.

For every $G \in \mathbb{G}$, we extend the function $f^G : \omega^2 \to \omega + 1$ from Definition 4.3 to a function $\bar{f}^G : \omega^2 + 1 \to \omega + 1$, letting $\bar{f}^G(\infty) = \omega$. Using the definition of f^G, one easily sees that for every finite set $F \subseteq \omega + 1$ of natural numbers, i.e., of isolated points, the set $(\bar{f}^G)^{-1}(F)$ is clopen in $\omega^2 + 1$ as a finite set of isolated points. This clearly implies that $\bar{f}^G$ is continuous.

The following lemma simplifies the task of proving that both $G \mapsto f^G$ and $G \mapsto \bar{f}^G$ actually yield continuous reductions.

Lemma 4.6. *The restriction map $C(\omega^2 + 1, \omega + 1) \to C(\omega^2, \omega + 1)$, $f \mapsto f|_{\omega^2}$ is a continuous one-to-one reduction with respect to embeddability.*

Proof. To check the continuity, let $K \subseteq \omega^2$ be compact, and let $U \subseteq \omega + 1$ be open. Then K is also compact as a subset of $\omega^2 + 1$, so for all $f \in C(\omega^2 + 1, \omega + 1)$,

$$f(K) \subseteq U \iff f|_{\omega^2}(K) \subseteq U,$$

which proves continuity.

It is one to one since ω^2 is a dense subset of $\omega^2 + 1$, and if two continuous functions agree on a dense subset, then they are equal.

To see that it is a reduction, notice that if $\sigma' : \omega^2 + 1 \to \omega^2 + 1$ is an embedding, then necessarily $\sigma'(\omega^2) = \omega^2$, so it restricts to an embedding $\sigma : \omega^2 \to \omega^2$. Moreover, if $\sigma : \omega^2 \to \omega^2$ is an embedding, then it extends to an embedding $\sigma' : \omega^2 + 1 \to \omega^2 + 1$ by letting $\sigma'(\omega^2) = \omega^2$. It follows that for every $f, g \in C(\omega^2 + 1, \omega + 1)$, σ embeds f in g if and only if σ' embeds $f|_{\omega^2}$ in $g|_{\omega^2}$. □

Remark 4.7. Notice that the restriction map of Lemma 4.6 is not a topological embedding of $C(\omega^2 + 1, \omega + 1)$ in $C(\omega^2, \omega + 1)$. Indeed, one can check that the direct image of the open set $S_{\omega^2+1,\omega+1}(\omega^2 + 1, \omega + 1 \setminus \{0\})$ is not open in $C(\omega^2, \omega + 1)$.

We now conclude our two special cases before attacking the general case.

Proposition 4.8. *The maps $G \mapsto f^G$ and $G \mapsto \bar{f}^G$ are continuous reductions from $(\mathbb{G}, \preccurlyeq)$ to $(C(\omega^2, \omega + 1), \sqsubseteq)$ and $(C(\omega^2 + 1, \omega + 1), \sqsubseteq)$, respectively.*

Proof. We claim that it only remains to prove the continuity of $G \mapsto \bar{f}^G$ from $\mathbb{G}$ to $C(\omega^2 + 1, \omega + 1)$ endowed with the compact-open topology in order to conclude. Assuming this fact for now, notice that for every $G \in \mathbb{G}$ we have $\bar{f}^G|_{\omega^2} = f^G$, so by Lemma 4.6, $G \mapsto \bar{f}^G \mapsto \bar{f}^G|_{\omega^2} = f^G$ is continuous as a composition of continuous maps, and it is a reduction by Lemmas 4.2, 4.4, and 4.5. But then by Lemma 4.6, $f^G \sqsubseteq f^H$ iff $\bar{f}^G \sqsubseteq \bar{f}^H$ for every $G, H \in \mathbb{G}$, so it follows that $G \mapsto \bar{f}^G$ is also a continuous reduction, as desired.

It remains for us to show that $G \mapsto \bar{f}^G$ is continuous. Fix compatible ultrametrics d on $\omega+1$ and d_G on the space of graphs. Since ω^2+1 is compact, the compact-open topology coincides with the uniform convergence topology. So it suffices to check that for every $\varepsilon > 0$ there exists a $\delta > 0$ so that whenever we have $d_G(G,H) < \delta$ for graphs G, H, then for every $x \in \omega^2+1$ we have $d(\bar{f}^G(x), \bar{f}^H(x)) < \varepsilon$.

To this end, fix $\varepsilon > 0$, and notice that the set $\{k \in \omega : d(k,\omega) \geq \varepsilon\}$ is finite and that whenever k and k' do not belong to this set, then $d(k,k') < \varepsilon$, as d is an ultrametric. Set

$$F = \{(m, \langle n, p\rangle_0) \mid \exists i\ (d(\langle i, \langle m,n\rangle_i, p\rangle_2, \omega) \geq \varepsilon)\}.$$

Then F is a finite set of isolated points in ω^2+1 and $F' = \{\{m,n\} \in [\omega]^2 \mid \exists p\, (m, \langle n,p\rangle_0) \in F\}$ is a finite set of pairs. Now we can find δ small enough such that $d_G(G,H) < \delta$ ensures that G and H agree on F'. By the definition of $\bar{f}^G$, this easily implies that $\bar{f}^G|_F = \bar{f}^H|_F$. Moreover, if $x \notin F$, then $d(\bar{f}^G(x), \bar{f}^H(x)) < \varepsilon$. □

4.2. **The general case.** Building on the previous section, we show that if X is Polish 0-dimensional with infinitely many points, then $(\mathbb{G}, \preccurlyeq)$ continuously reduces to $C(X, \omega+1)$. To do so, we will find a continuous map $\pi : X \to \omega^2+1$ such that $G \mapsto \bar{f}^G \circ \pi$ is the desired continuous reduction.

We start by identifying a sufficient condition for a map $\pi : X \to \omega^2+1$ to yield such a reduction.

Definition 4.9. Let us say that a pseudo-embedding $\rho : X \to \omega^2+1$ is *regular* if

(1) $|\rho^{-1}(\{\infty\})| \leq 1$;
(2) for every $m \in \mathbb{N}$, $|\rho^{-1}(\{(m,\omega)\})| = 1$;
(3) for every isolated points $y, y' \in \omega^2+1$, $\rho^{-1}(\{y\}) \neq \emptyset$ and $\rho^{-1}(\{y\})$ embeds in $\rho^{-1}(\{y'\})$.

Lemma 4.10. *Suppose that $\rho : X \to \omega^2+1$ is a regular pseudo-embedding. Then for every $G, H \in \mathbb{G}$, we have*

$$G \preccurlyeq H \quad \text{if and only if} \quad \bar{f}^G \circ \rho \sqsubseteq \bar{f}^H \circ \rho.$$

Proof. Let us start by remarking that for every $G \in \mathbb{G}$, the definition of the function $\bar{f}^G$ is such that y is a limit in ω^2+1 iff $\bar{f}^G(y) = \omega$.

We start with the forward implication. Assume that $G \preccurlyeq H$ so that there exists (σ, τ) which embeds $\bar{f}^G$ in $\bar{f}^H$ by Proposition 4.8. We need to check that y is isolated in ω^2+1 iff $\sigma(y)$ isolated in ω^2+1. This is not true for any embedding σ, but as (σ,τ) embeds $\bar{f}^G$ in $\bar{f}^H$, this follows from our first remark and from the fact that $\tau(\omega) = \omega$ holds. Moreover, σ being an embedding, it must fix the point ∞ (see Fact 2.3). It therefore follows from our hypotheses on ρ that $\rho^{-1}(\{y\})$ embeds in $\rho^{-1}(\{\sigma(y)\})$ for all $y \in \omega^2+1$. Since ρ is a pseudo-embedding, it follows from Proposition 3.7 that there exists some embedding $\bar{\sigma}$ such that $\rho \circ \bar{\sigma} = \sigma \circ \rho$. Hence $(\bar{\sigma}, \tau)$ embeds $\bar{f}^G \circ \rho$ in $\bar{f}^H \circ \rho$, as desired.

For the backward implication, assume that $(\bar{\sigma}, \tau)$ embeds $\bar{f}^G \circ \rho$ in $\bar{f}^H \circ \rho$ for some $G, H \in \mathbb{G}$. Let $L = \{x \in X \mid \rho(x) \text{ is the limit in } \omega^2+1\}$. We start by noticing that $x \in L$ iff $\bar{\sigma}(x) \in L$ for all $x \in X$. By our first remark, we have $x \in L$ iff $\bar{f}^G \circ \rho(x) = \omega$, and something similar for $\bar{f}^H$. So as $(\bar{\sigma}, \tau)$ embeds $\bar{f}^G \circ \rho$ in $\bar{f}^H \circ \rho$, we must have $x \in L$ iff $\bar{\sigma}(x) \in L$.

Notice that as ρ is a regular pseudo-embedding, ρ actually restricts to embedding from L in the limit points of ω^2+1. Let $x_{(m,\omega)} \in X$ denote the unique point with

$\rho(x_{(m,\omega)}) = (m,\omega)$, and let $x_\infty \in X$ be the unique point such that $\rho(x_\infty) = \infty$ (if there exists one at all). The set L is equal to either $\{x_{(m,\omega)} \mid m \in \mathbb{N}\}$ or to $\{x_\infty\} \cup \{x_{(m,\omega)} \mid m \in \mathbb{N}\}$. Since $\bar{\sigma}$ is restricted to an embedding of L in itself, it must fix the point x_∞ if it exists. In both cases, $\bar{\sigma}$ induces an injective map $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\bar{\sigma}(x_{(m,\omega)}) = x_{(\sigma(m),\omega)}$ for every m.

Next we prove that this injective map $\sigma : \mathbb{N} \to \mathbb{N}$ is a homomorphism from G to H. Assume that $n\ E^G\ m$. We claim that whenever $(x_k)_k$ is an injective sequence of isolated points in $\omega^2 + 1$ converging to (n,ω), then by possibly going to a subsequence, there exists a sequence $(x'_k)_k$ in X with $\rho(x'_k) = x_k$ for all k and such that $\rho(\bar{\sigma}(x'_k))_k$ is injective and converges to $(\sigma(n),\omega)$.

To see this, pick any sequence $(x'_k)_k$ in X with $\rho(x'_k) = x_k$. Since ρ is a pseudo-embedding and (x'_k) is not ρ-trivial, we have $x'_k \to x_{(n,\omega)}$. Now since $\rho(x'_k) = x_k$ is isolated in ω^2+1, so is $\rho(\bar{\sigma}(x'_k))$, and by continuity the sequence $\rho(\bar{\sigma}(x'_k))_k$ converges to the limit point $(\sigma(n),\omega)$. Therefore, by possibly going to a subsequence, we can assume that $\rho(\bar{\sigma}(x'_k))_k$ is injective, which proves the claim.

Since $n\ E^G\ m$, by definition of f^G there are injective sequences of isolated points $(x_k)_k$ and $(y_k)_k$ with $x_k \to (n,\omega)$, $y_k \to (m,\omega)$, and $\bar{f}^G(x_k) = \bar{f}^G(y_k)$ for all k. By possibly going simultaneously to subsequences, we use the claim to pick $(x'_k)_k$ and $(y'_k)_k$ in X such that $\rho(x'_k) = x_k$, $\rho(y'_k) = y_k$, and setting $x''_k = \rho \circ \bar{\sigma}(x'_k)$ and $y''_k = \rho \circ \bar{\sigma}(y'_k)_k$, $(x''_k)_k$ and $(y''_k)_k$ are injective and converge to $(\sigma(n),\omega)$ and $(\sigma(m),\omega)$, respectively. Moreover, for all k,

$$f^H(x''_k) = \tau \circ \bar{f}^G \circ \rho(x'_k) = \tau \circ \bar{f}^G \circ \rho(y'_k) = f^H(y''_k).$$

Therefore, using Lemma 4.5, we see that the two sequences $(x''_k)_k$ and $(y''_k)_k$ together witness that $\sigma(n)\ E^H\ \sigma(m)$ holds. □

If $(C_n)_n$ is a sequence of mutually disjoint clopen sets in a space X and $x \in X$, we write $C_n \to x$ if every neighborhood of X contains C_n for all but finitely many n. Notice that in this case, we have $x \notin C_n$ for all n, the set $F = \{x\} \cup \bigcup_n C_n$ is closed in X, and the map $\rho : F \to \omega + 1$ is given by $\rho(y) = n$ if $y \in C_n$ and if $\rho(x) = \omega$ is a pseudo-embedding.

Lemma 4.11. *Let X be an uncountable Polish 0-dimensional space. Then there exists a regular pseudo-embedding $\rho : X \to \omega^2 + 1$.*

Proof. Fix some compatible metric for X, and recall that a point $x \in X$ is a *condensation point* of X if every neighborhood of x is uncountable. Remember that X is uncountable if an only if the set of condensation points of X is uncountable [Kec12, 6.4, p. 32]. We fix some condensation point x_∞ of X and define a sequence $(C_m)_{m\in\omega}$ of pairwise disjoint clopen sets such that

(1) each C_m is uncountable,
(2) $C_m \to x_\infty$, and
(3) $X = \{x_\infty\} \cup \bigcup_m C_m$.

To this end, let $(V_n)_n$ be a decreasing sequence of clopen neighborhoods of x_∞ with vanishing diameter and $V_0 = X$. We claim that there exists a strictly increasing sequence $(n_i)_i$ of indices such that the clopen sets given by $C_0 = X \setminus V_{n_0}$ and $C_{i+1} = V_{n_i} \setminus V_{n_{i+1}}$ for $i \geq 0$ are all uncountable. This sequence (C_m) is clearly as desired. To see this, observe that there exists a condensation point $y \in V_0$ such that $y \neq x_\infty$, so $y \notin V_{n_0}$ for some large enough n_0. Then $X \setminus V_{n_0}$ is uncountable as a clopen neighborhood of the condensation point y. Next if n_i is defined, we can

apply the same argument inside V_{n_i} to get some $n_{i+1} > n_i$ such that $V_{n_i} \setminus V_{n_{i+1}}$ is uncountable, which completes the induction.

We then pick a condensation point $x_{(m,\omega)}$ in C_m for every m and apply the same construction to C_m and $x_{(m,\omega)}$ as we did above for X and x_∞. Hence we get for every m a sequence $(C_{m,n})_n$ of uncountable clopen sets that partitions $C_m \setminus \{x_{(m,\omega)}\}$ and satisfies $C_{m,n} \to x_{(m,\omega)}$.

We define $\rho : X \to \omega^2 + 1$ by $\rho(x_\infty) = \infty$, $\rho(x_{(m,\omega)}) = (m, \omega)$ and $\rho(x) = (m, n)$ if $x \in C_{m,n}$. Observe that whenever $(x_n)_n$ in X is not ρ-trivial, $\rho(x_n)_n$ can converge only to some limit point y of $\omega^2 + 1$, and by construction this happens if and only if $x_n \to x$ for the unique point x with $\rho(x) = y$. Hence ρ is a pseudo-embedding. To see that it is regular, notice that it clearly satisfies items 1 and 2 of Definition 4.9. To see condition 3, remember that every 0-dimensional Polish space embeds in 2^ω (see [Kec12, (7.8)]), while 2^ω embeds in every uncountable Polish space [Kec12, (6.5)]. □

Lemma 4.12. *Let X be a countable 0-dimensional space with infinitely many limit points. We can write X as the union of two closed sets F_0 and F_1 with the following properties:*

(1) *There is a regular pseudo-embedding $\rho : F_0 \to \omega^2 + 1$.*
(2) $F_0 \cap F_1 = \rho^{-1}(\{\infty\})$.

Proof. We start by making the following remark. If x is a limit point which is isolated in $\mathrm{CB}_1(X)$, then for every neighborhood U of x there exists a clopen set C with $x \in C \subseteq U$, $C \cap \mathrm{CB}_1(X) = \{x\}$, and such that C is homeomorphic to either $\omega+1$ or $[\omega]^2_+$. To see the last point, notice that if C is clopen with $C \cap \mathrm{CB}_1(X) = \{x\}$, then C is a Polish space with a single limit point, and by Proposition 2.4 it is homeomorphic to one of the three spaces $\omega \sqcup \omega + 1$, $\omega + 1$, $[\omega]^2_+$, and by shrinking C we can actually make sure that it is homeomorphic to either $\omega + 1$ or $[\omega]^2_+$.

Since X is countable with infinitely many limit points, $\mathrm{CB}_1(X)$ is infinite and countable. It follows that there are infinitely many isolated points in $\mathrm{CB}_1(X)$, and we can take an injective sequence $(x_n)_n$ of isolated points in $\mathrm{CB}_1(X)$. By going to a subsequence, we can assume that either $(x_n)_n$ converges to some x_∞ in X or that no subsequence of $(x_n)_n$ converges in X.

First, suppose that no subsequence of $(x_n)_n$ converges in X. Consider mutually disjoint clopen neighborhoods U_n of each point x_n. Applying our first remark inductively, we get a sequence $(C_n)_n$ of mutually disjoint clopen sets with $C_n \cap \mathrm{CB}_1(X) = \{x_n\}$ and vanishing diameters for some compatible metric on X. Then by possibly going to a subsequence, we can assume that either $C_n \cong \omega + 1$ for all n or $C_n \cong [\omega]^2_+$ for all n. We set $F_0 = \bigcup_n C_n$ and $F_1 = X \setminus \bigcup_n C_n$. Since the C_n have vanishing diameters and no subsequence of $(x_n)_n$ converges in X, it follows that F_0 is also closed. We define $\rho : F_0 \to \omega^2 + 1$ as follows. In the first case, we identify F_0 with $\bigsqcup_n \omega + 1$ and take ρ to be the inclusion into $\omega^2 + 1$. In the second case, notice that the continuous map $\pi : [\omega]^2_+ \to \omega + 1$ given by $\pi(\{k, l\}) = k$ and $\pi(\emptyset) = \omega$ is such that $|\pi^{-1}(\{\omega\})| = 1$ and that $\pi^{-1}(\{k\})$ is homeomorphic to ω for all k. Therefore, identifying F_0 with $\bigsqcup_n [\omega]^2_+$, we simply let $\rho = \bigsqcup_n \pi$.

Suppose now that $x_n \to x_\infty$. For some compatible metric on X, take mutually disjoint clopen neighborhoods U_n of each x_n whose diameters converge to 0. This last condition ensures that $U_n \to x_\infty$. By applying our first remark and possibly going to a subsequence, we can find a sequence $(C_n)_n$ of clopen sets with $C_n \subseteq U_n$,

$C_n \cap \mathrm{CB}_1(X) = \{x_n\}$, and either $C_n \cong \omega + 1$ for all n or $C_n \cong [\omega]^2_+$ for all n. We set $F_0 = \{x_\infty\} \cup \bigcup_n C_n$ and $F_1 = X \setminus \bigcup_n C_n$. Notice that F_0 is closed since each C_n is clopen and $C_n \to x_\infty$. In the first case, we can take ρ to be a homeomorphism from F_0 to $\omega^2 + 1$. In the second case, let $h_n : C_n \to [\omega]^2_+$ be a homeomorphism. We define $\rho : F_0 \to \omega^2 + 1$ by $\rho(x) = (n, \pi \circ h_n(x))$ if $x \in C_n$ and $\rho(x_\infty) = \infty$.

In all cases, the proof that ρ is a pseudo-embedding is just as in Lemma 4.11. Whenever $(x_n)_n$ in F_0 is not ρ-trivial, then $\rho(x_n)_n$ can converge only to some limit point y of $\omega^2 + 1$ and, by construction, this happens if and only if $x_n \to x$ for the unique point x in F_0 with $\rho(x) = y$. Moreover, by construction, ρ is regular in all cases. □

We are now ready to prove the main result of this section.

Theorem 4.13. *If X is a Polish 0-dimensional space with infinitely many limit points and Y is not discrete, then $(\mathbb{G}, \preccurlyeq)$ continuously reduces to $(C(X,Y), \sqsubseteq)$.*

Proof. As Y is not discrete, $\omega+1$ embeds in Y, so by Proposition 3.4 it is enough to show the result for $Y = \omega + 1$. Our continuous reduction $\mathbb{G} \to C(X, \omega+1)$ is given by $G \mapsto \bar{f}^G \circ \pi$ for some well chosen $\pi : X \to \omega^2 + 1$. We start by noticing that as long as π is continuous, then the map $C(\omega^2 + 1, \omega + 1) \to C(X, \omega + 1)$, $f \mapsto f \circ \pi$ is continuous. Suppose that $K \subseteq X$ is compact, and suppose that $U \subseteq \omega + 1$ is open. Then the set of $f \in C(\omega^2 + 1, \omega + 1)$ such that $f(\pi(K)) \subseteq U$ is open in $C(\omega^2+1, \omega+1)$ since $\pi(K)$ is compact. Since we proved that $G \mapsto \bar{f}^G$ is continuous in Proposition 4.8, the map $G \mapsto \bar{f}^G \circ \pi$ is continuous from $\mathbb{G}$ to $C(X, \omega + 1)$.

If X is uncountable, we simply take π to be the regular pseudo-embedding ρ given by Lemma 4.11, and it follows from Lemma 4.10 that $G \preccurlyeq H$ iff $\bar{f}^G \circ \pi \sqsubseteq \bar{f}^H \circ \pi$. So we indeed have a reduction in this case.

If X is countable, Lemma 4.12 allows us to write X as the union of two closed sets F_0 and F_1 and grants us with the existence of a regular pseudo-embedding $\rho : F_0 \to \omega^2 + 1$. Moreover, either $F_0 \cap F_1 = \emptyset$ or $F_0 \cap F_1 = \{x_\infty\}$ for the unique point of F_0 such that $\rho(x_\infty) = \infty$. We take $\pi : X \to \omega^2 + 1$ to be the extension of ρ which sends every point in F_1 to ∞. Notice that π is continuous: if F_0 and F_1 are disjoint, π is continuous as the topological sum of two continuous functions. In the other case, the proof is similar to the one we gave for Lemma 2.1.

Let us check that this π indeed yields a reduction from $\preccurlyeq$ to $\sqsubseteq$. Suppose that $G \preccurlyeq H$. By Lemma 4.10, there is some embedding $\bar{\sigma} : F_0 \to F_0$ which embeds $\bar{f}^G \circ \rho$ in $\bar{f}^H \circ \rho$. Then we can extend $\bar{\sigma}$ to an embedding σ of the whole space X by letting σ be the identity on F_1 (using Lemma 2.1 if F_0 and F_1 are not disjoint). Clearly σ embeds $\bar{f}^G \circ \pi$ in $\bar{f}^H \circ \pi$, as desired.

Conversely, assume that $\sigma : X \to X$ embeds $\bar{f}^G \circ \pi$ in $\bar{f}^H \circ \pi$. We claim that σ is restricted to an embedding $\bar{\sigma} : F_0 \to F_0$. This clearly implies that $\bar{\sigma}$ embeds $\bar{f}^G \circ \rho$ in $\bar{f}^H \circ \rho$, so $G \preccurlyeq H$ by Lemma 4.10. To see the claim, notice that for $I = \{x \in X \mid \rho(x) \text{ is isolated in } \omega^2 + 1\}$ we have $x \in I$ iff $\sigma(x) \in I$. Moreover, F_0 is equal to the closure of I. It follows that $x \in F_0$ iff $\sigma(x) \in F_0$, as desired. □

Remark 4.14. As we have seen, the space $C(X,Y)$ is not Polish in general. There is, however, another way of looking at the reductions given in Theorem 4.13 by considering continuous *partial* functions in the codes. As X and Y are Polish 0-dimensional, they are homeomorphic to the spaces of infinite branch through pruned trees S and T, respectively. Then every partial continuous function φ^* from X to Y with Polish domain is coded by some monotone function $\varphi : S \to T$ [Kec12, (2.6)]. The

space of monotone functions from S to T is a closed subset of the Polish space T^S, and hence Polish. Note that the continuous total functions form a coanalytic subset of this space. It is not hard to see that we can define in a continuous fashion for every $G \in \mathbb{G}$ a monotone map $\varphi^G : S \to T$ which codes the continuous function $: X \to Y$ given for G by our reduction.

Corollary 4.15. *Let X and Y be Polish 0-dimensional with X compact. If X has infinitely many limit points and Y is not discrete, then the restriction of the quasi-order of embeddability to $C(X,Y)$ is analytic complete.*

Proof. We proved in Theorem 3.2 that the quasi-order of embeddability is analytic on $C(X,Y)$. By Theorem 4.13, we have a continuous reduction from $(\mathbb{G}, \preccurlyeq)$ to $(C(X,Y), \sqsubseteq)$. Hence this follows from the result of Louveau and Rosendal (Fact 2.2). □

5. Order

We now discuss the other half of the dichotomy and prove that the restriction of the embeddability quasi-order is in several cases a well-quasi-order. To do so, we appeal to a technical strengthening of the notion of well-quasi-order called better-quasi-order (BQO), introduced by Nash-Williams [NW65].

5.1. **For a fistful of better-quasi-orders.** For the reader's convenience, we give here some basic definitions and collect some results on BQOs which are needed in the sequel. An introduction to the theory is given in [Sim85, Peq17], while further results can be found, for example, in [Mar94, vMS87, LSR90, CP14].

We let $[\omega]^\omega$ denote the set of infinite subsets of ω with the topology induced by the topology on 2^ω under the identification of a set with its characteristic function. For $X \in [\omega]^\omega$, we let $[X]^\omega$ be the set of infinite subsets of X endowed with the induced topology. If Q is a set, a *Q-multi-sequence* is a function f with domain $[X_0]^\omega$ for some $X_0 \in [\omega]^\omega$ and range a countable subset of Q. A Q-multi-sequence f is locally constant if $f^{-1}(\{q\})$ is open for every $q \in Q$. When $\leq_Q$ is a quasi-order on Q, we say that a Q-multi-sequence f is *bad* if for all $X \in \mathrm{dom}(f)$ we have $f(X) \not\leq_Q f(X^+)$, where $X^+ = X \setminus \{\min X\}$. A quasi-order on Q is a *BQO* if there are no bad locally constant Q-multi-sequences.

Recall that a quasi-order Q is WQO if and only if there is no *bad sequence* in Q, i.e., no sequence $(q_n)_n$ with $q_m \not\leq q_n$ whenever $m < n$. Observe now that any bad sequence $(q_n)_n$ in Q induces a locally constant bad Q-multi-sequence given by $f(X) = q_{\min X}$. Hence every BQO is indeed a WQO. Straightforward applications of the Galvin–Prikry theorem [GP73] show that every finite sum or finite product of BQOs (and in particular any finite QO) is a BQO.

We say that a map $\sigma : Q \to P$ is a *cohomomorphism* if for all p, q in Q, $\sigma(p) \leq_P \sigma(q)$ implies that $p \leq_Q q$. Notice that if there exists a cohomomorphism from Q to P and P is BQO, then Q is a BQO too.

We will use a result due to van Engelen, Miller, and Steel [vMS87], which establishes a very strong form of Fraïssé's conjecture and refines the result of Laver [Lav71]. To state this important result, we need to introduce the following strengthening of the BQO property for classes of structures.

Definition 5.1 ([LSR90])**.** Let $\mathcal{C}$ be a class of structures together with $\mathcal{C}$-morphisms between them. Assume that all identity maps are $\mathcal{C}$-morphisms, and assume that $\mathcal{C}$-morphisms are closed under composition.[3] Given a QO Q, the class $Q^{\mathcal{C}}$ of *Q-labeled $\mathcal{C}$-structures* is given by

$$Q^{\mathcal{C}} = \{f \mid f \text{ is a function, } \operatorname{dom}(f) \text{ is a } \mathcal{C}\text{-structure, } \operatorname{im}(f) \subseteq Q\},$$

together with the following quasi-order:

$$f_0 \leq f_1 \leftrightarrow \exists \mathcal{C}\text{-morphism } g : \operatorname{dom}(f_0) \to \operatorname{dom}(f_1) \text{ such that } \forall x \in \operatorname{dom}(f_0)\ f_0(x) \leq_Q f_1(g(x)).$$

We say that $\mathcal{C}$ *preserves* BQOs if for all BQO Q the class $Q^{\mathcal{C}}$ is also BQO.

Notice that if $\mathcal{C}$ is a class of structures, then the $\mathcal{C}$-morphisms induce a quasi-order where $C \leq C'$ for two structures C and C' in $\mathcal{C}$ if and only if there exists a $\mathcal{C}$-morphism from C to C'. Of course, if a class $\mathcal{C}$ preserves BQOs, then in particular $\mathcal{C}$ is a BQO under this quasi-order, as one easily sees by considering the one-point BQO.

The class Lin_{c} consists of the countable linear orders. Given two linear orders K and L, a Lin_{c}-morphism is an order preserving injection (or equivalently, an order embedding) from K to L that is continuous with respect to the order topology.

Theorem 5.2 ([vMS87, Theorem 3.5])**.** *The class Lin_{c} preserves BQOs.*

Remark 5.3. In the same line of research as the results presented in this paper, Camerlo and Marcone studied the complexity of the quasi-order of embeddability on the class of Q-labeled linear orders for some countable partial order Q. After Laver's [Lav71] proof that this is a BQO when Q is a BQO, Camerlo [Cam05] proved that it is an analytic complete quasi-order when Q is not WQO. The exact complexity of embeddability when Q is WQO but not BQO is still unknown, but see [CM07].

Let us denote by $\mathcal{Q}$ the class consisting of the single linear order $\mathbb{Q}$ together with topological embeddings as $\mathcal{Q}$-morphisms. Notice that a map between two linear orders is order preserving and injective if and only if it is an embedding of linear orders. Moreover, if $j : K \to L$ is an order embedding between linear orders equipped with the order topology, then the direct image of any open set in K is relatively open in the range of j. Therefore, any Lin_{c}-morphism $f : \mathbb{Q} \to \mathbb{Q}$ is in particular a topological embedding. We state the following direct corollary of Theorem 5.2 for later use.

Corollary 5.4. *The class $\mathcal{Q}$ preserves BQOs.*

We denote by $P0D$ the class of Polish 0-dimensional spaces quasi-ordered by topological embeddability. We recall [Car13, Theorem 1.3] the following.

Theorem 5.5. *The quasi-order $P0D$ is a BQO.*

In the sequel, if Q is a quasi-order, then we denote by $Q_\star$ the quasi-order obtained from Q by adding a new element, $\star$, which is not comparable to any other element of Q. Of course, Q is BQO if and only if $Q_\star$ is BQO.

[3] A class of structures as understood here corresponds exactly to a concrete category in the category theory sense.

5.2. **For a few well-quasi-orders more.** Since any countable metrizable space is homeomorphic to a closed subspace of $\mathbb{Q}$ (see [Kec12, (7.12)]), we have the following.

Fact 5.6. For every continuous function f with a 0-dimensional Polish domain X and a countable image, there is an $f^{\mathbb{Q}} \in C(X, \mathbb{Q})$ with $f \sqsubseteq f^{\mathbb{Q}}$ and $f^{\mathbb{Q}} \sqsubseteq f$.

We first show that when X is discrete, then embeddability is a BQO on $C(X, Y)$. Since every continuous function from a discrete space is locally constant, this follows from the following more general result.

Theorem 5.7. *The class LC of locally constant functions with Polish 0-dimensional domain is better-quasi-ordered by topological embeddability.*

Proof. Let $LC_{\mathbb{Q}}$ denote the class of locally constant functions $f : X \to \mathbb{Q}$, where X is Polish 0-dimensional quasi-ordered by embeddability. By Fact 5.6, it suffices to show that $LC_{\mathbb{Q}}$ is BQO.

We define for every $f \in LC_{\mathbb{Q}}$ a function $\Lambda_f : \mathbb{Q} \to P0D_\star$ with the following property: whenever $\tau : \mathbb{Q} \to \mathbb{Q}$ witnesses $\Lambda_f \leq \Lambda_g$ in $P0D_\star^{\mathcal{Q}}$, we can find an embedding $\sigma : X \to X$ such that (σ, τ) is a witness for $f \sqsubseteq g$. Combining Corollary 5.4 and Theorem 5.5, we get $P0D_\star^{\mathcal{Q}}$ to be a BQO. This makes $f \mapsto \Lambda_f$ a cohomomorphism, which implies that $(LC_{\mathbb{Q}}, \sqsubseteq)$ is BQO.

Given $f : X \to \mathbb{Q}$, let

$$\Lambda_f : \mathbb{Q} \longrightarrow P0D_\star,$$
$$q \longmapsto \begin{cases} f^{-1}(\{q\}) & \text{if } q \in \operatorname{im} f, \\ \star & \text{otherwise.} \end{cases}$$

Notice that for all $q \in \operatorname{im} f$, the set $f^{-1}(\{q\})$ is a clopen subset of X, and as such, it is a Polish 0-dimensional space.

Take $f : X \to \mathbb{Q}$ and $g : Y \to \mathbb{Q}$ as locally constant, with X and Y Polish 0-dimensional, and suppose that $\Lambda_f \leq \Lambda_g$ holds in $P0D_\star^{\mathcal{Q}}$, as witnessed by some embedding $\tau : \mathbb{Q} \to \mathbb{Q}$.

By the role of $\star$, we have $q \in \operatorname{im} f$ if and only if $\tau(q) \in \operatorname{im} g$ for all $q \in Q$, so τ is restricted to an embedding : $\operatorname{im} f \to \operatorname{im} g$. Moreover, for every $q \in \operatorname{im} f$, we can pick an embedding $\sigma_q : f^{-1}(\{q\}) \to g^{-1}(\{\tau(q)\})$. As f and g are locally constant, $f^{-1}(\{q\})$ is clopen in X and $g^{-1}(\{\tau(q)\})$ is clopen in Y. This easily implies that $\bigcup_{q \in \operatorname{im} f} \sigma_q$ is an embedding of X in Y. As $\tau \circ f = g \circ \sigma$ clearly holds, we get $f \sqsubseteq g$ via (τ, σ), as desired. □

Theorem 5.8. *If X is a locally compact Polish 0-dimensional space with finitely many nonisolated points, then $(C(X, Y), \sqsubseteq)$ is a BQO.*

Proof. We start by reducing the problem in several steps. Any Polish space with only finitely many limit points is in particular countable, so any function with such a domain has a countable image. So once again it is enough to treat the case of rational valued functions. We prove that topological embeddability on $C(X, \mathbb{Q})$ is a BQO.

If D is a discrete space, then every continuous function from D is locally constant. Therefore, as in the proof[4] of Theorem 5.7, we have map $C(D, \mathbb{Q}) \to P0D_\star^{\mathcal{Q}}$, $f \mapsto \Lambda_f$

[4]As D is discrete, any injective function $\sigma : D \to D$ is an embedding, so we do not really need $P0D$ in this case. We could use $\omega + 1$ instead and let $\Lambda_f(q) = |f^{-1}(\{q\})|$ if $q \in \operatorname{im} f$, and $\star$ otherwise.

such that if τ witnesses $\Lambda_f \leq \Lambda_g$ in $P0D^{\mathcal{Q}}_\star$, then $f \sqsubseteq g$, as witnessed by (σ, τ) for some embedding $\sigma : D \to D$.

We will define a BQO L and associate with every $f \in C(\omega+1, \mathbb{Q})$ a function $\Gamma_f : \mathbb{Q} \to \mathrm{L}$ such that if τ witnesses $\Gamma_f \leq \Gamma_g$ in $\mathrm{L}^{\mathcal{Q}}$, then $f \sqsubseteq g$ via (σ, τ) for some σ.

Assuming this for now, we show that this enough to conclude. Suppose first that X has a unique limit point. Applying Proposition 2.4 and using the fact that X is locally compact, we have $X = D \sqcup \omega + 1$ for some (possibly empty) discrete space D. Consider the map $\Gamma^X : C(X, \mathbb{Q}) \to (P0D_\star \times \mathrm{L})^{\mathcal{Q}}$ given by $\Gamma^X(f)(q) = (\Lambda_{f_0}(q), \Gamma_{f_1}(q))$ for every $q \in \mathbb{Q}$, where $f_0 = f|_D$ and $f_1 = f|_{\omega+1}$. Then, as finite products of BQOs are BQO, $(P0D_\star \times \mathrm{L})^{\mathcal{Q}}$ is BQO by Corollary 5.4. Moreover, Γ^X is also a cohomomorphism. To see this, notice that if τ witnesses $\Gamma^X(f) \leq \Gamma^X(g)$ in $(P0D_\star \times \mathrm{L})^{\mathcal{Q}}$, then there exist two embeddings $\sigma_0 : D \to D$ and $\sigma_1 : \omega + 1 \to \omega + 1$ such that (σ_i, τ) embeds f_i in g_i for $i = 0, 1$. Hence clearly $(\sigma_0 \cup \sigma_1, \tau)$ embeds f in g.

Now suppose that X is locally compact with $n+1$ limit points. Then by Fact 2.3, $X = \bigsqcup_{i \leq n} X_i$, where each X_i has a single limit point and is locally compact too. Consider the map $\Gamma^X : C(X, \mathbb{Q}) \to (\prod_{i \leq n}(P0D_\star \times \mathrm{L}))^{\mathcal{Q}}$ given by $\Gamma^X(f)(q) = (\Gamma^{X_i}(f_i)(q))_{i \leq n}$ for every $q \in \mathbb{Q}$ where $f_i = f|_{X_i}$. As before, $(\prod_{i \leq n}(P0D_\star \times \mathrm{L}))^{\mathcal{Q}}$ is BQO, and the proof that Γ^X is a cohomomorphism is similar.

We take L to be the disjoint union $(\omega+1) \sqcup \omega \sqcup \{\star\}$ quasi-ordered by $(i, p) \leq (j, q)$ if and only if $i = j$ and $p \leq q$. Then L is BQO as a finite sum of BQOs. Let $f : \omega + 1 \to \mathbb{Q}$ be continuous. Notice that if $q \neq f(\omega)$, then by continuity of f the set $f^{-1}(\{q\})$ must be finite. Define

$$\Gamma_f : \mathbb{Q} \longrightarrow \mathrm{L},$$
$$q \longmapsto \begin{cases} (0, |f^{-1}(\{q\})|) & \text{if } f(\omega) = q, \\ (1, |f^{-1}(\{q\})|) & \text{if } f(\omega) \neq q, \text{ but } q \in \operatorname{im} f, \\ (2, \star) & \text{otherwise.} \end{cases}$$

Suppose that $\tau : \mathbb{Q} \to \mathbb{Q}$ is an embedding witnessing $\Gamma_f \leq \Gamma_g$. By the role of $\star$, τ is necessarily restricted to an embedding : $\operatorname{im} f \to \operatorname{im} g$. Moreover, we must have $\tau(f(\omega)) = g(\omega)$, and the cardinality of $f^{-1}(\{f(\omega)\})$ is lower or equal to that of $g^{-1}(\{g(\omega)\})$. Therefore, taking any injection $\sigma_\omega : f^{-1}(\{f(\omega)\}) \to g^{-1}(\{g(\omega)\})$ with $\sigma_\omega(\omega) = \omega$, we have an embedding. Finally, for every $q \in \operatorname{im} f$ with $q \neq f(\omega)$, there is an injection $\sigma_q : f^{-1}(\{q\}) \to g^{-1}(\{\tau(q)\})$. By taking the union of all of the σ_q together with σ_ω, we obtain an injection $\sigma : \omega + 1 \to \omega + 1$ such that $\sigma(\omega) = \omega$; therefore, σ is an embedding. Since clearly $g \circ \sigma = \tau \circ f$, it follows that (σ, τ) witnesses $f \sqsubseteq g$. □

6. Maximal elements in Baire classes

In this section, we investigate the existence of maximal elements for embeddability in the Baire classes. We start by looking at the very specific case of the class of continuous functions.

6.1. **Continuous functions.** Consider the Hilbert cube $[0, 1]^\omega$, and let proj : $[0, 1]^\omega \times [0, 1]^\omega \to [0, 1]^\omega$ be the projection $\mathrm{proj}(\beta, \gamma) = \gamma$.

Proposition 6.1. *The map* proj *is a maximum for the embeddability quasi-order on continuous functions between separable metrizable spaces. In particular, it is a maximum on continuous functions between Polish spaces.*

Proof. Let $f : X \to Y$ be a continuous function between two separable metrizable spaces. The Hilbert cube is universal for separable metrizable spaces ([Kec12, (4.14)]), so there exist embeddings $\sigma' : X \to [0,1]^\omega$ and $\tau : Y \to [0,1]^\omega$. Define $\sigma : X \to [0,1]^\omega \times [0,1]^\omega$ by $\sigma(x) = (\sigma'(x), \tau(f(x))$. As f is continuous and X, Y are separable metrizable, one easily sees that σ is an embedding. Moreover, we have $\mathrm{proj}(\sigma(x)) = \tau(f(x))$ for all $x \in X$. Hence (σ, τ) embeds f in proj, as desired. □

We remark that using the universal property of the Baire space (see [Kec12, (7.8)]), the same argument shows that the corresponding projection $\mathrm{proj}' : \omega^\omega \times \omega^\omega \to \omega^\omega$ is a maximum for embeddability on continuous functions between metrizable 0-dimensional spaces.

We briefly mention that while the class of continuous functions admits a maximum, the class of discontinuous ones admits a two-element basis. Let us call $d_0 : \omega + 1 \to 2$ the characteristic function of $\{\omega\}$, and let $d_1 : \omega + 1 \to \omega$ denote the bijection given by $d_1(\omega) = 0$ and $d_1(n) = n + 1$.

Proposition 6.2. *Let $f : X \to Y$ be any function between separable metrizable spaces. Then exactly one of the following two holds:*

(1) $f \sqsubseteq \mathrm{proj}$ *or*
(2) *either* $d_0 \sqsubseteq f$ *or* $d_1 \sqsubseteq f$.

Moreover, proj, d_0, *and* d_1 *form an antichain.*

Proof. Let us first note that proj, d_0, and d_1 are pairwise incomparable. As their images have distinct cardinalities, we have $\mathrm{proj} \not\sqsubseteq d_0$, $\mathrm{proj} \not\sqsubseteq d_1$, and $d_1 \not\sqsubseteq d_0$. Since proj is continuous while d_0 and d_1 are not, it follows that $d_0 \not\sqsubseteq \mathrm{proj}$ and $d_1 \not\sqsubseteq \mathrm{proj}$. Finally, as d_1 is injective and d_0 is not, we get $d_0 \not\sqsubseteq d_1$.

Since proj is maximal among continuous functions between separable metrizable spaces by Proposition 6.1, it remains to see that any discontinuous function $f : X \to Y$ between separable metrizable spaces embeds either d_0 or d_1. So suppose that $f : X \to Y$ is not continuous and take a sequence $(x_n)_n$ in X such that $x_n \to x$ but $f(x_n) \not\to f(x)$ in Y. By passing to a subsequence, we can assume that for all $n \in \mathbb{N}$ we have $x_n \neq x$, and since $x_n \to x$, by going to a further subsequence we can also assume that $(x_n)_n$ is injective. We distinguish two cases. First, suppose that f takes only finitely many values on $(x_n)_n$. Then we can go to a subsequence on which f is constant and we set $\tau_0 : 2 \to Y$, $\tau_0(0) = f(x_0)$, and $\tau_0(1) = f(x)$. Otherwise, f takes infinitely many values on $(x_n)_n$ and we can find a subsequence on which f is injective. In this case, we define $\tau_1 : \omega \to Y$ by $\tau_1(0) = f(x)$ and $\tau_1(n+1) = f(n)$. In both cases, we let $\sigma : \omega + 1 \to Y$ be given by $\sigma(n) = x_n$ and $\sigma(\omega) = x$. In the first case, (σ, τ_0) embeds d_0 in f, and in the second case, (σ, τ_1) embeds d_1 in f. □

6.2. **Ranks and embeddability.** For separable metrizable spaces X and Y and a countable ordinal $\xi \geq 1$, recall that a function $f : X \to Y$ is Baire class ξ if the preimage by f of any open subset of Y is $\mathbf{\Sigma}^0_{\xi+1}$ in X ([Kec12, 24.3]). We let $\mathcal{B}_\xi(X, Y)$ denote the class of Baire class ξ functions from X to Y, and we simply write $\mathcal{B}_\xi(X)$ when $X = Y$. In contrast with the class of continuous functions, we prove in this subsection that there are no greatest elements with respect to $\sqsubseteq$ in

the classes of functions $\mathcal{B}_\xi(X,Y)$ if $1 \le \xi < \omega_1$, if X is an uncountable Polish space and Y is an arbitrary Polish space with $|Y| \ge 2$. Moreover, if X is a 0-dimensional Polish space or a euclidean space, then we further show that there are no maximal elements in $\mathcal{B}_\xi(X)$ for $2 \le \xi < \omega_1$. This will be proved using a modified version of the ranks defined on the class $\mathcal{B}_1$ by Bourgain [Bou78], Zalcwasser [Zal30], Gillespie and Hurwicz [GH30], and others, and extensively studied by Kechris and Louveau [KL90]. The theory of these ranks has been generalized to higher Baire classes in [EKV16, Kis17, EV17]. Usually these ranks are defined solely for real valued functions, but in our investigation we define a rank in a more general context resembling the so-called separation rank of Bourgain.

We fix for the rest of this section two Polish spaces X and Y with X uncountable and $|Y| \ge 2$.

Definition 6.3. If A is a $\mathbf{\Delta}^0_{\xi+1}$ subset of X, then it can be expressed as a transfinite difference of a countable decreasing sequence of $\mathbf{\Pi}^0_\xi$ sets, i.e.,

$$A = \bigcup_{\eta<\lambda,\eta \text{ even}} F_\eta \setminus F_{\eta+1},$$

where $F_\eta \in \mathbf{\Pi}^0_\xi$ (see [Kec12, 22.E]). We call the minimal length of such a sequence the *rank* of A, and we denote it by $\rho_{\xi,0}(A)$.

If A_1 and A_2 are disjoint $\mathbf{\Pi}^0_{\xi+1}$ sets, they can be separated by a $\mathbf{\Delta}^0_{\xi+1}$ set; i.e., there is a $\mathbf{\Delta}^0_{\xi+1}$ set A so that $A_1 \subseteq A \subseteq X \setminus A_2$. We denote by $\rho_{\xi,1}(A_1,A_2)$ the minimal rank of such a separating set.

From this definition and the Hausdorf–Kuratowski analysis (see [Kec12, Theorem 22.27]), we get the following.

Fact 6.4. For each disjoint $A_1, A_2 \in \mathbf{\Pi}^0_{\xi+1}$, the rank $\rho_{\xi,1}(A_1,A_2) < \omega_1$. It is well known that the rank $\rho_{\xi,0}$ is unbounded in ω_1 on the class $\mathbf{\Delta}^0_{\xi+1}(X)$ (see, e.g., [EKV16, Theorem 4.3]). Hence for every ordinal $\alpha < \omega_1$ taking sets A and $X \setminus A$ such that $A \in \mathbf{\Delta}^0_{\xi+1}$ and $\rho_{\xi,0}(A) > \alpha$, we have $\rho_{\xi,1}(A, X \setminus A) > \alpha$.

Definition 6.5. Let $f \in \mathcal{B}_\xi(X,Y)$. Define

$$\rho_\xi(f) = \sup_{y_1,y_2 \in Y \text{ distinct}} \rho_{\xi,1}(f^{-1}(y_1), f^{-1}(y_2)).$$

Notice that this definition makes sense: if $f \in \mathcal{B}_\xi(X,Y)$, then the preimages $f^{-1}(y_1), f^{-1}(y_2)$ are disjoint $\mathbf{\Pi}^0_{\xi+1}(X)$ sets. Moreover, the following easy claims show that this rank is meaningful on the class $\mathcal{B}_\xi(X,Y)$.

Lemma 6.6. *If $f \in \mathcal{B}_\xi(X,Y)$, then $\rho_\xi(f) < \omega_1$.*

Proof. Let $\mathcal{B} = \{U_n : n \in \omega\}$ be a countable basis in Y. Define

$$\rho'_\xi(f) = \sup\{\rho_{\xi,1}(f^{-1}(\overline{U}), f^{-1}(\overline{V})) : U, V \in \mathcal{B}, \overline{U} \cap \overline{V} = \emptyset\}.$$

Since $f \in \mathcal{B}_\xi(X,Y)$, if $\overline{U} \cap \overline{V} = \emptyset$, then the sets $f^{-1}(\overline{U}), f^{-1}(\overline{V})$ are disjoint $\mathbf{\Pi}^0_{\xi+1}(X)$ sets, so the supremum is taken over a countable set of countable ordinals. Therefore, $\rho'_\xi(f) < \omega_1$.

We show that $\rho_\xi(f) \le \rho'_\xi(f)$, which is enough to prove the claim. Indeed, for every $y_1, y_2 \in Y$ distinct, there exist $U, V \in \mathcal{B}$ with $\overline{U} \cap \overline{V} = \emptyset$ and $y_1 \in U$, $y_2 \in V$. But then if for a $\mathbf{\Delta}^0_{\xi+1}(X)$ set A we have $f^{-1}(\overline{U}) \subseteq A \subseteq X \setminus f^{-1}(\overline{V})$, then of

course $f^{-1}(y_1) \subseteq A \subseteq X \setminus f^{-1}(y_2)$. Thus, by definition, $\rho_{\xi,1}(f^{-1}(y_1), f^{-1}(y_2)) \leq \rho_{\xi,1}(f^{-1}(\overline{U}), f^{-1}(\overline{V}))$, and in turn $\rho_\xi(f) \leq \rho'_\xi(f)$. □

Lemma 6.7. *We have*

$$\sup\{\rho_\xi(f) \mid f \in \mathcal{B}_\xi(X,Y)\} = \omega_1,$$

in fact, the rank of functions with range of cardinality at most 2 *is unbounded in* ω_1.

Proof. By Fact 6.4, for every $\alpha < \omega_1$, there exists a $\mathbf{\Delta}^0_{\xi+1}(X)$ set A such that $\rho_{\xi,1}(A, X \setminus A) > \alpha$. Choosing $y_1, y_2 \in Y$ distinct and letting

$$f(x) = \begin{cases} y_1 & \text{if } x \in A, \\ y_2 & \text{otherwise,} \end{cases}$$

we have $\rho_\xi(f) = \rho_{\xi,1}(A, X \setminus A) > \alpha$. □

Proposition 6.8. *If* $g \in \mathcal{B}_\xi(X,Y)$, $f \in \mathcal{B}_\xi(X',Y')$, *and* $g \sqsubseteq f$, *then* $\rho_\xi(g) \leq \rho_\xi(f)$.

Proof. Let σ and τ witness the embedding $g \sqsubseteq f$. Let $y_1, y_2 \in Y$ be distinct. It is enough to prove that $\rho_{\xi,1}(g^{-1}(y_1), g^{-1}(y_2)) \leq \rho_\xi(f)$. Since τ is an embedding, $f^{-1}(\tau(y_1))$ and $f^{-1}(\tau(y_2))$ are disjoint and there exists a set $A \in \mathbf{\Delta}^0_{\xi+1}(X')$ separating them such that $\rho_{\xi,0}(A) \leq \rho_\xi(f)$. Therefore, there exists a sequence $(F_\eta)_{\eta<\rho_\xi(f)}$ of $\mathbf{\Pi}^0_\xi(X')$ sets with

$$A = \bigcup_{\eta<\rho_\xi(f),\eta \text{ even}} F_\eta \setminus F_{\eta+1}.$$

Then clearly

$$\sigma^{-1}(A) = \bigcup_{\eta<\rho_\xi(f),\eta \text{ even}} \sigma^{-1}(F_\eta) \setminus \sigma^{-1}(F_{\eta+1}),$$

and since σ is a topological embedding, the sequence $(\sigma^{-1}(F_\eta))_{\eta<\rho_\xi(f)}$ is a decreasing sequence of $\mathbf{\Pi}^0_\xi(X)$ sets, so $\rho_{\xi,0}(\sigma^{-1}(A)) \leq \rho_\xi(f)$. From $\tau \circ g = f \circ \sigma$, we get

$$\begin{aligned} x \in g^{-1}(y_1) &\longleftrightarrow \tau(g(x)) = \tau(y_1) \\ &\longleftrightarrow f(\sigma(x)) = \tau(y_1) \longleftrightarrow x \in \sigma^{-1}(f^{-1}(\tau(y_1))), \end{aligned}$$

so $g^{-1}(y_1) = \sigma^{-1}(f^{-1}(\tau(y_1)))$. But, since A separates $f^{-1}(\tau(y_1))$ and $f^{-1}(\tau(y_2))$, we have $\sigma^{-1}(A)$ separating $\sigma^{-1}(f^{-1}(\tau(y_1))) = g^{-1}(y_1)$ and $\sigma^{-1}(f^{-1}(\tau(y_2))) = g^{-1}(y_2)$. Thus, $\rho_{\xi,1}(g^{-1}(y_1), g^{-1}(y_2)) \leq \rho_{\xi,0}(\sigma^{-1}(A)) \leq \rho_{\xi,0}(A) \leq \rho_\xi(f)$, which finishes the proof. □

As an immediate corollary, we get the following weaker version of the main theorem of this section.

Corollary 6.9. *There is no* $\sqsubseteq$*-greatest element among Baire class* ξ *functions from* $X \to Y$, *i.e.,* $f \in \mathcal{B}_\xi(X,Y)$ *and* $g \sqsubseteq f$ *for every* $g \in \mathcal{B}_\xi(X,Y)$.

Proof. By contradiction, suppose that f is a $\sqsubseteq$-greatest element in $\mathcal{B}_\xi(X,Y)$. By Lemma 6.6, the rank of f is some countable ordinal α. By Lemma 6.7, there exists a Baire class ξ function $g : X \to Y$ whose rank is some countable ordinal β strictly bigger than α. Now, as f is maximal, we have $g \sqsubseteq f$, so by Proposition 6.8 we have $\beta \leq \alpha$, which is a contradiction. □

Now, we are ready to prove the main theorem of this section; it implies in particular Theorem 1.4.

Theorem 6.10. *Let $1 \leq \xi < \omega_1$, let X, Y be Polish spaces such that there exists an embedding $h : X \to X$, with $|X \setminus h(X)| = \mathfrak{c}$ and $h(X)$ being $\mathbf{\Delta}^0_2$, and $|Y| \geq 2$. There is no $\sqsubseteq$-maximal function in $\mathcal{B}_\xi(X, Y)$. In particular, this is true if X is an uncountable 0-dimensional Polish space or a euclidean space.*

Proof. Let f be an arbitrary function in $\mathcal{B}_\xi(X, Y)$. Take a perfect set $P \subseteq X \setminus h(X)$. As in Lemma 6.7, take a $\mathbf{\Delta}^0_{\xi+1}(P)$ set A with $\rho_{\xi,0}(A) > \rho_\xi(f)$, and define a function $g_0 : P \to Y$ by picking two distinct points y_1 and y_2 in Y and letting $g_0(x) = y_1$ if $x \in A$, and $g_0(x) = y_2$ otherwise. Let $y \in Y$ be arbitrary, and define

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in P, \\ f \circ h^{-1}(x) & \text{if } x \in \operatorname{im} h, \\ y & \text{otherwise.} \end{cases}$$

It is easy to see that g is a Baire-ξ function: for an open $U \subseteq Y$, we have $g^{-1}(U)$ being the union of at most three sets from the collection $h(f^{-1}(U)), P, X \setminus (P \cup h(A)), A, P \setminus A$. Clearly, using the facts that $\operatorname{im} h$ is $\mathbf{\Delta}^0_2$ and h is a homeomorphism, we get all of these sets to be $\mathbf{\Sigma}^0_{\xi+1}$.

On the one hand, $g \circ h = f \circ h^{-1} \circ h = f$, so $f \sqsubseteq g$. On the other hand, if a set B separates $g^{-1}(y_1)$ from $g^{-1}(y_2)$, then $B \cap P = A$. By the fact that P is closed, intersecting each element of a transfinite sequence of $\mathbf{\Pi}^0_{\xi+1}$ sets in X with P, we still get such a sequence with at most the same length. Hence if we can express B as a union of differences of a sequence of length α, then this gives an expression of A in P with a union of the differences of a sequence of length at most α. Thus, by the definition of our rank, $\rho_\xi(g) \geq \rho_{\xi,0}(A) > \rho_\xi(f)$. Then by Proposition 6.8, we get $g \not\sqsubseteq f$, so f is not maximal. □

Remark 6.11. It is not hard to see that Theorem 6.10 cannot be extended to arbitrary Polish spaces: let X be a so-called Cook continuum [Coo67]. Then every continuous nonconstant function from X to X is the identity (see [GO03]). Hence if we consider the functions from X to X, then the embedding relation trivializes; that is, $g \sqsubseteq f$ if and only if $f = g$. Thus, every function is maximal.

7. Concluding remarks and open questions

Combining Theorems 4.13, 5.7, and 5.8, we have obtained a dichotomy for the quasi-order of embeddability on $C(X, Y)$ when X is locally compact:

(1) either it continuously reduces an analytic complete quasi-order or
(2) it is a better-quasi-order.

First, when X is locally compact and not compact, we do not know in general the exact complexity of the quasi-order of embeddability on the Polish space $C(X, Y)$. While a direct calculation gives a $\mathbf{\Sigma}^1_2$ definition, there are examples where embeddability is not analytic. We now briefly explain this fact. Notice that a function $f : X \to X$ is an embedding if and only if $f \sqsubseteq \mathrm{id}_X$. Hence if $\sqsubseteq$ is analytic on $C(X, X)$, then so is the set of embeddings of X in itself.

Proposition 7.1. *The set of embeddings of $\mathbb{N} \sqcup 2^\omega$ in itself is coanalytic complete for the compact-open topology.*

Proof. We give a continuous reduction from the well-founded trees on ω to the set of embeddings. For any tree T on ω, we define a continuous function $f_T : \omega^{<\omega} \sqcup 2^\omega \to \omega^{<\omega} \sqcup 3^\omega$ as follows. If $s = (s_0, \dots, s_n) \in \omega^{<\omega}$, then $f_T(s) = s$ if $s \notin T$ and $f_T(s) = 0^{s_0}10^{s_1}1 \cdots 0^{s_n}12^\omega$ if $s \in T$. If $\alpha \in 2^\omega$, then $f_T(\alpha) = \alpha$. Clearly f_T is continuous and injective for all tree T. Suppose that T has an infinite branch γ. Then $(\gamma|_n)_n$ does not converge in the domain, while $f_T(\gamma|_n)_n$ converges to some $f_T(\alpha)$ in the range. Hence f_T is not an embedding. When T is well founded, then f_T is easily seen to be an embedding.

To see that the map $T \mapsto f_T$ is continuous, notice that any compact subset K of $\omega^{<\omega} \sqcup 2^\omega$ contains only finitely many points in $\omega^{<\omega}$. Therefore, the condition $f_T(K) \subseteq U$ for some open set U depends only on a finite piece of T. □

It follows that embeddability is not analytic on $C(\mathbb{N} \sqcup 2^\omega, \mathbb{N} \sqcup 2^\omega)$. Moreover, if Y is uncountable and not compact, then $\mathbb{N} \sqcup 2^\omega$ embeds in Y. Therefore, by Proposition 3.4, it follows that embeddability is not analytic on $C(\mathbb{N} \sqcup 2^\omega, Y)$ either. By Theorem 4.13, embeddability is a $\mathbf{\Sigma}^1_1$-hard quasi-order, but we do not know its exact complexity.

Second, we believe that one can drop the local compactness assumption on X in the above dichotomy. From the proof of Theorem 5.8, it appears that an essential step in this direction is to prove the following conjecture.

Conjecture 7.2. *The quasi-order* $(C([\omega]^2_+, \mathbb{Q}), \sqsubseteq)$ *is BQO.*

Let us now make a remark on embeddability beyond the case of continuous functions. A direct application of [vMS87, Theorem 3.3] shows that when X is a compact 0-dimensional Polish space and Y is finite, the Baire class 1 functions from X to Y are BQO under embeddability. This begs for us to consider the following much broader problem.

Question 7.3. If X has only finitely many limit points, or if Y is discrete, are the Baire class 1 functions from X to Y better-quasi-ordered by embeddability? What about all Borel functions from X to Y?

In a different direction, recall that a *basis* for some upward-closed subset S of a quasi-order Q is a subset $B \subseteq S$ such that $q \in S$ iff $\exists b \in B\ b \leq q$. As a matter of fact, a quasi-order is a well-quasi-order if and only if every upward-closed subset admits a finite basis. As we proved that embeddability on $C(X, Y)$ is in most cases very complicated and far from being a well-quasi-order, this shows the existence of many upward-closed sets of continuous functions with no finite basis. This contrasts drastically with the existence of finite bases for some important classes, such as the discontinuous functions (Proposition 6.2), the functions that are not σ-continuous with closed witnesses [Sol98, CM17a], the non-σ-continuous functions [PS12], and the non-Baire class 1 functions [CM17a]. This motivates us to make a conjecture and formulate a general problem.

Conjecture 7.4. *For all* $\alpha < \omega_1$*, the class of functions that are not Baire class* α *admit a finite basis for topological embeddability.*

Question 7.5. Which classes of functions admit a finite basis?

Finally, the situation described by Proposition 6.2 suggests an even more specific question.

Question 7.6. Are there, continuous functions aside, other examples of a class of functions that both are defined as the downward closure of finitely many functions with respect to embeddability and whose complements have a finite basis?

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