Sia $f : U \longrightarrow V$ $\quad U \subseteq \mathbb{R}^n$ aperto (1)
$\quad V \subseteq \mathbb{R}^m$ aperto

f di classe $\mathcal{C}^\infty$

ricordiamo $\Omega^0(U) = \mathcal{C}^\infty(U) = \{g : U \to \mathbb{R} \mid g\ \mathcal{C}^\infty\}$

$\Omega^0(V) = \mathcal{C}^\infty(V) = \{h : V \to \mathbb{R} \mid h\ \mathcal{C}^\infty\}$

f induce una funzione:

$$f^* : \mathcal{C}^\infty(V) \longrightarrow \mathcal{C}^\infty(U)$$

$$g \longrightarrow g \circ f$$

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}$$

$f^*(g)(p) =$

$= g(f(p))$

si scrive $\quad f^*(g) = g \circ f$

E' facile vedere che f^* è un omomorfismo, cioè:

$$f^*(g+h) = f^*(g) + f^*(h)$$

$$f^*(g \cdot h) = f^*(g) \cdot f^*(h)$$

(esercizio)

f^* si chiama pull back

vogliamo definire un omomorfismo analogo per le forme differenziali

$f : U \longrightarrow V$ ②

$\rightsquigarrow \quad f^* : \Omega^k(V) \longrightarrow \Omega^k(U)$

<u>Definizione</u> sia $\omega \in \Omega^k(V)$

$f^*\omega$ è la forma $\in \Omega^k(U)$ definita da:

$$\boxed{\begin{aligned} f^*\omega|_p(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k) &= \\ &= \omega|_{f(p)}(df(\underline{v}_1), \dots, df(\underline{v}_k)) \end{aligned}}$$

dove $p \in U$, $\underline{v}_1, \dots, \underline{v}_k \in T_pU = T_p\mathbb{R}^n$

<u>Esempio</u> $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

(x_1, x_2) coord di $\mathbb{R}^2$, (y_1, y_2, y_3) coord di $\mathbb{R}^3$

$f(x_1, x_2) = (2x_1x_2, x_1 + 2x_2, \sin(x_1) + \cos(x_2))$

$\omega \in \Omega^2(\mathbb{R}^3)$

$$\omega = y_1\, dy_1 \wedge dy_2 + y_3\, dy_2 \wedge dy_3$$

sia $p = (0,0) \in \mathbb{R}^2$

$f(p) = (0, 0, 1) \in \mathbb{R}^3$

$f^*\omega_{|p}(\underline{v}_1, \underline{v}_2) = ?$ (3)

poiché è multilineare, alternante basta calcolarla sui vettori della base $\{\underline{e}_1, \underline{e}_2\}$ di $\mathbb{R}^2_v$

$$f^*\omega_{|p}(\underline{e}_1, \underline{e}_1) = f^*\omega_{|p}(\underline{e}_2, \underline{e}_2) = 0$$

$\longrightarrow$ l'unico calcolo da fare è $f^*\omega_{|p}(\underline{e}_1, \underline{e}_2)$

$$f^*\omega_{|p}(\underline{e}_1, \underline{e}_2) = \omega_{|f(p)}(df(\underline{e}_1), df(\underline{e}_2))$$

$$\omega_{|f(p)} = dy_2 \wedge dy_3$$

$$(dy_2 \wedge dy_3)(df(\underline{e}_1), df(\underline{e}_2))$$

$$df(\underline{e}_1) = (f \circ \gamma_1)'(0) =$$

$\underline{e}_1$ = vett. tangente alla curva $\gamma_1(t) = (t, 0)$

$\underline{e}_2$ = ———————— $\gamma_2(t) = (0, t)$

si ha: $f \circ \gamma_1 = (0, t, \operatorname{sen} t)$

$f \circ \gamma_2 = (0, 2t, \cos t)$

$$df(\underline{e}_1) = (0, 1, 1) = \underline{f}_2 + \underline{f}_3$$
$$df(\underline{e}_2) = (0, 2, 0) = 2\underline{f}_2$$

(4)

$$f^*\omega|_p(\underline{e}_1, \underline{e}_2) = \omega|_{f(p)}(\underline{f}_2+\underline{f}_3, 2\underline{f}_2)$$

$$= (dy_2 \wedge dy_3)(f_2+f_3, 2f_2)$$

$$= \det \begin{bmatrix} dy_2(f_2+f_3) & dy_2(2f_2) \\ dy_3(f_2+f_3) & dy_3(2f_2) \end{bmatrix} =$$

$$= \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = -2$$

<u>Osservazione</u>: $f^*\omega|_p \in \Lambda^2 T_p^* \mathbb{R}^2_a$

e poiché $\dim T_p^* \mathbb{R}^2_a = 2$

si ha $\dim \Lambda^2 T_p^* \mathbb{R}^2_a = 1$

<u>base</u>: $\Lambda^2 T_p^* \mathbb{R}^2_a = \{ dx_1 \wedge dx_2 \}$

<u>quindi</u>: $f^*\omega|_p = \lambda (dx_1 \wedge dx_2)$

inoltre: $f^*\omega|_p(\underline{e}_1, \underline{e}_2) = -2$

$$(dx_1 \wedge dx_2)(\underline{e}_1, \underline{e}_2) = 1$$

$$\Longrightarrow \quad f^*\omega_p = -2(dx_1 \wedge dx_2)$$

$p = (0,0)$

Per capire meglio il pullback, cominciamo a vedere alcune proprietà (5)

Proposizione $f : U \longrightarrow V$

$\omega, \eta \in \Omega^k(V)$, $g \in \Omega^0(V)$ funzione

Allora:

(1) $f^*(\omega + \eta) = f^*(\omega) + f^*(\eta)$

(2) $f^*(g \cdot \omega) = f^*(g) \cdot f^*(\omega)$

(3) $\omega_1, \omega_2, \dots, \omega_k$ sono delle 1-forme allora:

$$f^*(\omega_1 \wedge \dots \wedge \omega_k) = f^*(\omega_1) \wedge \dots \wedge f^*(\omega_k)$$

Dim (1), (2) leggere sulle dispense

(3): $f^*(\omega_1 \wedge \dots \wedge \omega_k)(\underline{v}_1, \dots, \underline{v}_k) =$

$$= (\omega_1 \wedge \dots \wedge \omega_k)(df(\underline{v}_1), \dots, df(\underline{v}_k))$$

$$= \det\left(\omega_i(df(\underline{v}_j))\right)$$

$$= \det\left(f^*\omega_i(\underline{v}_j)\right)$$

$$= (f^*\omega_1 \wedge \dots \wedge f^*\omega_k)(\underline{v}_1, \dots, \underline{v}_k)$$

Il pullback si calcola mediante "sostituzione" 6

$(x_1, \dots, x_n)$ coord in $\mathbb{R}^n$, $U \subseteq \mathbb{R}^n$
$(y_1, \dots, y_m)$ coord in $\mathbb{R}^m$, $V \subseteq \mathbb{R}^m$

$f : U \to V$ differenziabile

$$y_1 = f_1(x_1, \dots, x_n), \ \dots, \ y_m = f_m(x_1, \dots, x_n)$$

$\omega \in \Omega^k(V)$ $\quad \omega = \sum_I a_I \cdot dy_I$

$$f^*\omega = f^*\left(\sum_I a_I \, dy_I\right)$$

$$= \sum_I f^*(a_I \, dy_I) \quad \text{per } 1$$

$$= \sum_I f^*(a_I) \cdot f^*(dy_I) \quad \text{per } 2$$

$$= \sum_I f^*(a_I) \cdot \left(f^*dy_{i_1} \wedge \dots \wedge f^*dy_{i_k}\right) \quad \text{per } 3$$

$$f^*dy_i(\underline{v}) = dy_i(df(\underline{v})) =$$

$$= d(y_i \circ f)(\underline{v}) = df_i(\underline{v})$$

cioè $\boxed{f^*dy_i = df_i}$

dunque:

$$f^*\omega = \sum_I a_I\big(f_1(x_1 \ldots x_n), \ldots, f_m(x_1 \ldots x_n)\big)\cdot$$
$$\cdot\, df_{i_1} \wedge \ldots \wedge df_{i_k}$$

cioè sostituiamo, $y_i = f_i$

$$dy_i = df_i$$

Esempio: $f(x_1, x_2) = (2x_1x_2,\ x_1 + 2x_2,\ \operatorname{sen} x_1 + \cos x_2)$

$$\omega = y_1\, dy_1 \wedge dy_2 + y_3\, dy_2 \wedge dy_3$$

$$dy_1 = df_1 = d(2x_1x_2) = 2x_2\, dx_1 + 2x_1\, dx_2$$
$$dy_2 = df_2 = dx_1 + 2dx_2$$
$$dy_3 = \cos x_1\, dx_1 - \operatorname{sen} x_2\, dx_2$$

$$\therefore f^*\omega = 2x_1x_2\,(2x_2\, dx_1 + 2x_1\, dx_2) \wedge (dx_1 + 2dx_2)$$
$$+ (\operatorname{sen} x_1 + \cos x_2)(dx_1 + 2dx_2) \wedge (\cos x_1\, dx_1 - \operatorname{sen} x_2\, dx_2)$$

$$= 2x_1x_2 \left(4x_2 \, dx_1 \wedge dx_2 + 2\, dx_2 \wedge dx_1 \right)$$ 8

$$+ (\operatorname{sen} x_1 + \cos x_2) \left(-\operatorname{sen} x_2 \, dx_1 \wedge dx_2 + 2\cos x_1 \, dx_2 \wedge dx_1 \right)$$

$$= \left[8x_1x_2^2 - 4x_1x_2 - (\operatorname{sen} x_1 + \cos x_2) \operatorname{sen} x_2 - 2(\operatorname{sen} x_1 + \cos x_2)\cos x_1 \right] dx_1 \wedge dx_2$$

$f^*\omega_p$ con $p = (0,0)$?

$$\longrightarrow -2(dx_1 \wedge dx_2)$$

Si ottiene adesso che f^* rispetta il prodotto wedge di forme qualunque:

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$$

$y_i = f_i(x_1, \dots, x_n)$ $\qquad \omega = \sum a_I \, dy_I$

$\eta = \sum b_J \, dy_J$

$$f^*(\omega \wedge \eta) = f^*\left(\sum_{I,J} a_I b_J \, dy_I \wedge dy_J \right)$$

$$= \sum_{I,J} a_I(f_i)\, b_J(f_i)\, df_I \wedge df_J$$

$$= \left(\sum_I a_I(f_i)\, df_I \right) \wedge \left(\sum_J b_J(f_i)\, df_J \right)$$

$= f^* \omega \wedge f^* \eta$ ∎

Conclusione: sia $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ aperti

$f : U \longrightarrow V$ di classe C^∞

Allora si ha indotto un omomorfismo

$$f^* : \Omega^*(V) \longrightarrow \Omega^*(U)$$

di anelli cioè
$$f^*(\omega + \eta) = f^*(\omega) + f^*(\eta)$$
$$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$$

f^* mantiene il grado cioè

$$\omega \in \Omega^k(V) \longrightarrow f^*\omega \in \Omega^k(U)$$

—o—

Sia $g \in C^\infty(U) = \Omega^0(U)$

$dg|_p \in T_p^* U$ è una forma lineare

variando $p \in U$ si ottiene $dg \in \Omega^1(U)$

una 1-forma differenziale

$$\boxed{dg_p = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(p) \cdot dx_i}$$

dunque $\quad d : \Omega^0(U) \to \Omega^1(U)$

$$g \to dg$$

- $d(f+g) = df + dg$
- $d(\lambda f) = \lambda\, df \qquad \lambda \in \mathbb{R}$
- $d(f \cdot g) = g\, df + f\, dg \qquad$ (Leibniz)

Vogliamo definire $d : \Omega^k(U) \to \Omega^{k+1}(U)$

Definizione: $\omega = \sum_I a_I\, dx_I \in \Omega^k(U)$

allora:

$$\boxed{d\omega = \sum_I da_I \wedge dx_I}$$

Esempio

$$\omega = x_3\, dx_1 + \sin x_1\, dx_2 + x_1 x_2\, dx_3$$

$$d\omega = dx_3 \wedge dx_1 + d(\sin x_1) \wedge dx_2 + d(x_1 x_2) \wedge dx_3$$

$$= -dx_1 \wedge dx_3 + \cos x_1\, dx_1 \wedge dx_2 + x_1\, dx_2 \wedge dx_3 + x_2\, dx_1 \wedge dx_3 =$$

$$= \cos x_1 \, dx_1 \wedge dx_2 + x_1 \, dx_2 \wedge dx_3 + (x_2 - 1) \, dx_1 \wedge dx_3$$

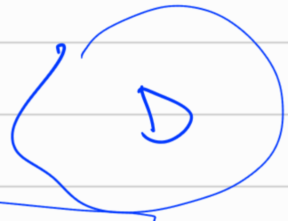
Esempio: $U = \mathbb{R}^2$

$$\omega = \boxed{P(x,y)\,dx + Q(x,y)\,dy}$$

$$d\omega = (P_x dx + P_y dy) \wedge dx + (Q_x dx + Q_y dy) \wedge dy =$$

$$= \boxed{(Q_x - P_y)\, dx \wedge dy}$$

Green dice



∂D = bordo

$$\boxed{\int_{\partial D} \omega = \iint_D d\omega}$$

Il teorema di Stokes generalizza questo enunciato

$d : \Omega^k(U) \to \Omega^{k+1}(U)$ si chiama

derivazione esterna

<u>Osservazione</u> ha senso calcolare

$$d^2 = d \circ d : \Omega^k(U) \to \Omega^{k+2}(U)$$

<u>Proposizione</u> $d^2 = 0$

<u>cioè</u>: $\forall \omega$ forma differenziale

$$d(d\omega) = 0$$

<u>Dimostrazione</u>

<u>passo 1</u> $\omega = g$ una funzione $\in \Omega^0(U)$

$$dg = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i$$

$$d(dg) = d\left(\sum_{i=1}^n \frac{\partial g}{\partial x_i} dx_i \right)$$

$$= \sum_{i=1}^n d\left(\frac{\partial g}{\partial x_i}\right) \wedge dx_i$$

$$= \sum_{i=1}^n \left[\left(\sum_{j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j} \right) dx_j \right] \wedge dx_i$$

$$= \sum_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j} dx_i \wedge dx_j$$

ora $i = j \to dx_i \wedge dx_i = 0$

se $i \neq j$ ho due termini nella somma (13)

$$\frac{\partial^2 g}{\partial x_i \partial x_j} dx_i \wedge dx_j + \frac{\partial^2 g}{\partial x_j \partial x_i} dx_j \wedge dx_i$$

$$= 0$$

Quindi la somma totale è 0.

<u>passo 2</u> ω qualunque

$$\omega = \sum' a_I dx_I$$

$$d\omega = \sum' da_I \wedge dx_I$$

$$d(d\omega) = d\left(\sum' da_I \wedge dx_I\right) =$$

$$= \sum' d(da_I \wedge dx_I)$$

<u>Osservazione</u> se ω, η sono forme

$$\boxed{d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta}$$

dove $k =$ grado di ω

$$d\left(\left(\sum' a_I dx_I\right) \wedge \left(\sum' b_J dx_J\right)\right) =$$

$$= d\left(\sum' a_I b_J dx_I \wedge d_J\right)$$

$$= \sum{}' d(a_I b_J) \wedge dx_I \wedge dx_J$$

$$= \sum{}' (a_I\, db_J + b_J\, da_I) \wedge dx_I \wedge dx_J$$

$$= \sum{}' b_J\, da_I \wedge dx_I \wedge dx_J$$

$$+ \sum{}' a_I\, db_J \wedge dx_I \wedge dx_J$$

$$= \sum{}' (da_I \wedge dx_I) \wedge b_J\, dx_J \longrightarrow d\omega \wedge \eta$$

$$+ (-1)^k \sum{}' a_I\, dx_I \wedge db_J \wedge dx_J \longrightarrow (-1)^k \omega \wedge d\eta$$

ritorniamo alla proposizione:

$$\sum{}' d(da_I \wedge dx_I) = \qquad dx_I = 1 \cdot dx_I$$

$$= \sum{}' \underbrace{d(da_I)}_{0} \wedge dx_I + da_I \wedge \underline{d(dx_I)}$$

$$= \sum{}' \quad 0 + da_I \wedge \underbrace{d(1)}_{=0} \cdot dx_I$$

$$= 0 \quad \blacksquare$$

Quindi: $\quad d^2\omega = 0 \quad \forall \omega$ forma differenziale

$$d(dx_I) = ?$$

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

$$d(\omega) = d\left(\sum a_I \, dx_I\right)$$

$$\underset{\text{defn.}}{=} \sum \underline{\underline{da_I}} \wedge \underline{\underline{dx_I}}$$

$dx_I = 1 \cdot dx_I$ wie $a_I = 1$

$$\rightarrow d(dx_I) = 0$$