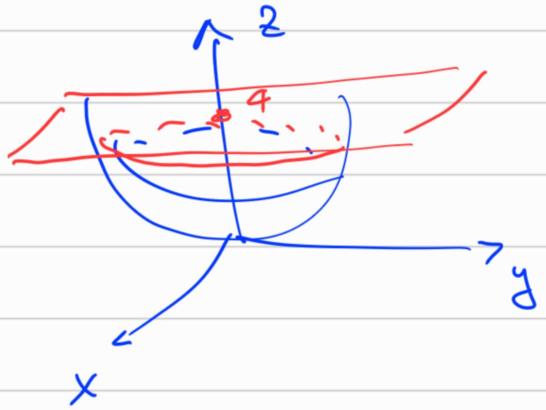
Esercizio 4.24 Abate Tovena p. 222

1

$S \subseteq \mathbb{R}^3 \qquad S = \{ z = x^2 + y^2 \}$

paraboloide di rotazione



(i) calcolare K

(ii) direzioni principali nei punti della curva

$\sigma : \mathbb{R} \to S$

$\sigma(t) = (2\cos t, 2\sin t, 4)$

direzioni principali di curvatura = autovettori di $-dN_p$

$-dN_p = I_p^{-1} \cdot II_p$

$\underline{X}(u,v) = (u, v, u^2+v^2)$ come grafico

$\underline{X}_u = (1, 0, 2u) \qquad \underline{X}_u \wedge \underline{X}_v = (-2u, -2v, 1)$

$\underline{X}_v = (0, 1, 2v)$

$E = \underline{X}_u \cdot \underline{X}_u = 1 + 4u^2$
$F = 4uv$
$G = 1 + 4v^2$

$I_p = \begin{bmatrix} 1+4u^2 & 4uv \\ 4uv & 1+4v^2 \end{bmatrix}$

$\underline{x}_{uu} = (0,0,2)$ $\underline{x}_{uv} = (0,0,0)$ $\underline{x}_{vv} = (0,0,2)$

2

$$\|\underline{x}_u \wedge \underline{x}_v\| = \sqrt{1+4u^2+4v^2} = \sqrt{EG-F^2}$$

$$N = \frac{1}{\sqrt{}}(-2u, -2v, 1)$$

$$e = \frac{2}{\sqrt{}}, \quad f = 0, \quad g = \frac{2}{\sqrt{}}$$

$$\mathrm{II}_p = \frac{1}{\sqrt{}}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$K \longrightarrow$ immediato $= \dfrac{eg-f^2}{EG-F^2} = \dots$

$$-dN_p = \mathrm{I}_p^{-1} \cdot \mathrm{II}_p$$

$$\mathbb{R} \longrightarrow \mathbb{R}^2 = U \xrightarrow{\underline{x}} S$$

$t \to (2\cos t, 2\sin t)$ $(u,v) \longrightarrow (u, v, u^2+v^2)$

$$\sigma = (2\cos t, 2\sin t, 4)$$

$$\begin{cases} u = 2\cos t \\ v = 2\sin t \end{cases}$$ in questi punti:

$$\mathrm{I}_p = \begin{bmatrix} 1+16\cos^2 t & 16\cos t\sin t \\ 16\cos t \sin t & 1+16\sin^2 t \end{bmatrix}$$

$$\det I_p = EG - F^2 = 1 + 16\cos^2 t + 16 \operatorname{sen}^2 t$$

3

$$= 17$$

$$I_p^{-1} = \frac{1}{17}\begin{bmatrix} 1+16\sin^2 t & -16\cos t \operatorname{sen} t \\ -16\cos t \operatorname{sen} t & 1+16\cos^2 t \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$II_p = \frac{1}{\sqrt{17}} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \frac{2}{\sqrt{17}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-dN_p = \frac{2}{17 \cdot \sqrt{17}}\begin{bmatrix} 1+16\sin^2 t & -16\operatorname{sen} t\cos t \\ -16\operatorname{sen} t\cos t & 1+16\cos^2 t \end{bmatrix}$$

autovettori??

$$\det\begin{bmatrix} (1+16\sin^2 t) - \lambda & -16\operatorname{sen} t\cos t \\ -16\operatorname{sen} t\cos t & (1+16\cos^2 t) - \lambda \end{bmatrix}$$

$$= (1+16\operatorname{sen}^2 t)(1+16\cos^2 t) - \lambda\left[1+16\cos^2 t + 1 + 16\operatorname{sen}^2 t\right]$$

$$+ \lambda^2 - 16 \cdot 16 \operatorname{sen}^2 t \cos^2 t$$

4

$$= \lambda^2 - \left[2 + \overset{16}{16\cos^2 t + 16\sin^2 t}\right] \lambda + (1 + \underset{16}{16\cos^2 t + 16\sin^2 t}) = 0$$

$$\boxed{\lambda^2 - 18\lambda + 17 = 0}$$

$$(\lambda - 1)(\lambda - 17) = 0 \qquad \boxed{\lambda = 1,\ 17}$$

$$-dN_p = \frac{2}{17\cdot\sqrt{17}} \begin{bmatrix} 1+16\sin^2 t & -16\sin t\cos t \\ -16\sin t\cos t & 1+16\cos^2 t \end{bmatrix}$$

aut vel $\lambda = 1$

$$\frac{32}{17\cdot\sqrt{17}} \begin{bmatrix} \sin^2 t & -\sin t\cos t \\ -\sin t\cos t & \cos^2 t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\sin^2 t \cdot x - \sin t\cos t\, y = 0$$

$$\sin t \left[\sin t\, x - \cos t\, y\right] = 0$$

$$t \neq 0,\ \neq \pi \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

direz. principale : $\cos t \cdot \underline{x}_u + \sin t \cdot \underline{x}_v$

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$$\begin{bmatrix} \cancel{1} + 16\sin^2 t - \cancel{17} - 16 & -16\sin t\cos t \\ -16\sin t\cos t & \cancel{1} + 16\cos^2 t - \cancel{17} - 16 \end{bmatrix}$$

$$= 16 \begin{bmatrix} \sin^2 t - 1 & -\sin t\cos t \\ -\sin t\cos t & \cos^2 t - 1 \end{bmatrix}$$

$$= -16 \begin{bmatrix} \cos^2 t & \sin t\cos t \\ \sin t\cos t & \sin^2 t \end{bmatrix}$$

. . . . .

— o —

Compito giugno 2017, Ex 1

$\sigma : I \to \mathbb{R}^3$ biregolare

$\sigma(I) \subseteq$ sfera di raggio $R > 0$

$k : I \to \mathbb{R}$ curvatura di σ

Allora: $k(t) \geq 1/R \quad \forall t$

- possiamo supporre $S : x^2 + y^2 + z^2 = R^2$ (cioè centro nell'origine)

- $\alpha : I \to S$ param x arcolunghezza
  $\alpha(s) = (x(s), y(s), z(s))$

Ipotesi: $\alpha(I) \subseteq S$

cioè: $\boxed{\alpha(s) \cdot \alpha(s) \equiv R^2}$

derivando:

$$2\,\alpha(s) \cdot \underline{\underline{\alpha'(s)}} \equiv 0$$

cioè: $\alpha(s) \perp \underline{t}(s)$

derivo ancora:

$$\alpha'(s) \cdot \alpha'(s) + \alpha(s) \cdot \alpha''(s) \equiv 0$$

$$1 + \big(\alpha(s) \cdot \underline{n}(s)\big)\, k(s) \equiv 0$$

$$\boxed{k(s) = -\frac{1}{\alpha(s) \cdot \underline{n}(s)}}$$

$$|\alpha(s) \cdot \underline{n}(s)| \le \|\alpha(s)\| \cdot \|\underline{n}(s)\| = \|\alpha(s)\| = R$$

poiché $k > 0$ per costruzione,

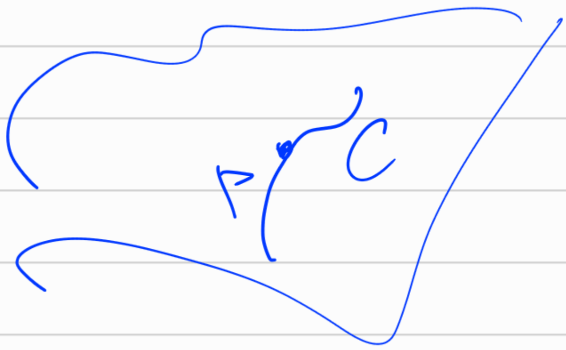
deve essere $\alpha(s) \cdot \underline{n}(s) < 0$

$$\longrightarrow \quad k = \left| -\frac{1}{\alpha(s) \cdot \underline{n}} \right| = \frac{1}{|\alpha(s) \cdot \underline{n}(s)|} \qquad \le R$$

$$\Longrightarrow \quad k \ge 1/R$$

Sapendo la teoria delle superfici

$$k_n = k \langle \underline{n}, \underline{N} \rangle$$



$\Longrightarrow |k_n| \leq k$

$k_n =$ curv. normale di C in P

però: ① k_n dipende non da C ma solo dalla direzione tg

② $k_1 \leq k_n \leq k_2$

curvature principali

Adesso:
- $k \geq |k_n|$
- su una sfera $k_n = 1/R$ in ogni punto perché $k_1 = k_2 = 1/R$

$\longrightarrow$ fine dell'esercizio.

Giugno 2018, Ex $\underline{\underline{2}}$ (8)

$S = \{x^4 + y^4 + z^4 = 1\}$

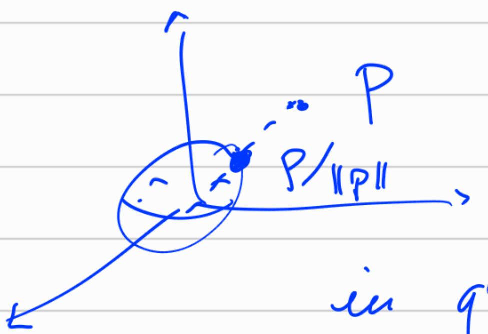
(1) S è reg e orientabile

(2) $F : S \to S^2$ (sfera unitaria)

$p \to p/\|p\|$

F è diffeomorfismo

- F è C^∞ perché è la restrizione di una funz. C^∞ su $\mathbb{R}^3 \setminus \{0\}$ e $S \subseteq \mathbb{R}^3 \setminus \{0\}$

- 

Domanda: una semiretta di vertice l'origine in quanti punti incontra S?

$r : (t x_0, t y_0, t z_0) \quad t > 0$

$r \cap S \qquad t^4 (x_0^4 + y_0^4 + z_0^4) = 1$

$t^4 = \frac{1}{(\quad)} \qquad t = \pm\sqrt{(\quad)}$

una sola radice positiva.

3) S è compatta + connessa

ovvio, omeomorfa alla sfera

$K : S \longrightarrow \mathbb{R}$ curvatura gaussiana di S

$$\int_S K \, dA = ?? = 4\pi$$

Teorema di Gauss-Bonnet

S connessa, compatta, orientabile

allora:

$$\int_S K \, dA = 2\pi \cdot \chi(S)$$

esempio: per $S = S^2$, $K \equiv 1/R^2$

$$\int_S \frac{1}{R^2} \, dA = \frac{1}{R^2} \cdot \text{area}(S)$$

$$= \frac{1}{R^2} \cdot 4\pi R^2 = 4\pi$$

$\sigma : \mathbb{R} \longrightarrow \mathbb{R}^3$

$$\sigma(t) = g(t) \cdot (\cos t, \sin t, 0)$$

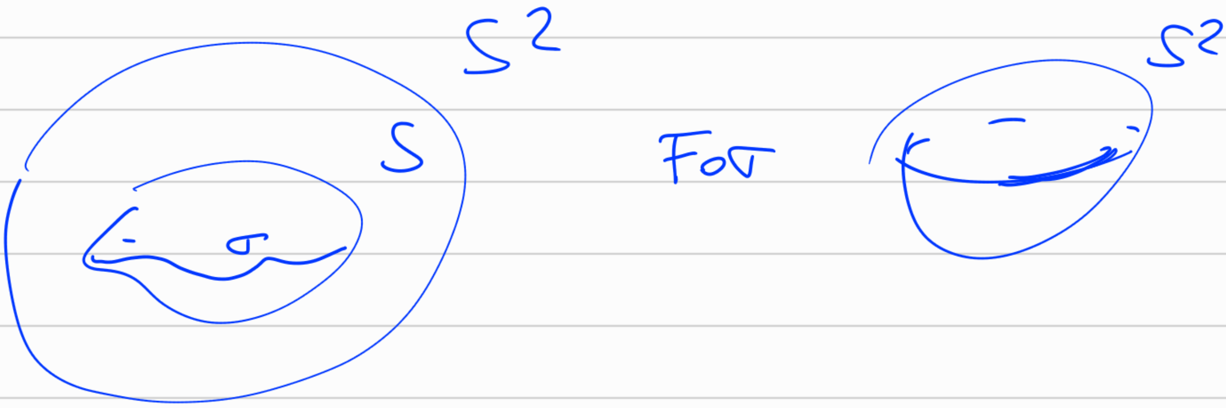
$$g(t) = \frac{1}{\sqrt[4]{\cos^4 t + \sin^4 t}}$$

σ = curva in $\mathbb{R}^3$

• $\sigma(\mathbb{R}) \subseteq S$ $\qquad x^4 + y^4 + z^4 = 1$

<u>ovvio</u>

• $\|\sigma'(\pi/4)\|$, • $\|(F \circ \sigma)'(\pi/4)\|$



$$\sigma(t) = g(t) \cdot (\cos t, \sin t, 0)$$

$$\sigma'(t) = (\cos t \cdot g'(t) - g(t) \cdot \sin t,$$
$$\sin t \cdot g'(t) + g(t) \cos t, \quad 0)$$

$$g'(t) = -\tfrac{1}{4} (\cos^4 t + \sin^4 t)^{-5/4} \cdot (-4\cos^3 t \sin t + 4 \sin^3 t \cos t)$$

$g'(\pi/4) =$

$\cos \pi/4 = \sin \pi/4 = \frac{\sqrt{2}}{2}$ $\quad (\;)^2 = 1/2 \, , \; (\;)^4 = 1/4$

$g'(\pi/4) = -1/4 \left(\frac{1}{4} + \frac{1}{4} \right)^{-5/4} \cdot 0 = 0$

$g(\pi/4) = \left(1/4 + 1/4 \right)^{-1/4} = \left(1/2 \right)^{-1/4} =$

$= 2^{1/4} = \sqrt[4]{2}$

$\sigma'(\pi/4) = \sqrt[4]{2} \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right)$

$= g(\pi/4) \underbrace{\left(-\sin \pi/4, \cos \pi/4, 0 \right)}_{\| \quad \| = 1}$

$$\boxed{\| \sigma'(\pi/4) \| = \sqrt[4]{2}}$$

$\sigma(t) = g(t) \cdot (\cos t, \sin t, 0)$

$(F \circ \sigma)(t) = \frac{\sigma(t)}{\| \sigma(t) \|} = (\cos t, \sin t, 0)$

$(F \circ \sigma)'(t) = (-\sin t, \cos t, 0)$

$$\boxed{\| (F \circ \sigma)'(\pi/4) \| = 1}$$

S sup data $\{z = 1 - x^2 - y^2\}$ 12

orientiamo S in modo che $\underline{N}_{\underset{\text{punto}}{(0,0,1)}} = \underset{\text{vettore}}{(0,0,1)}$

$S \to$ carta, oppure catena
cioè devo parametrizzarla.

$\underline{X} : \mathbb{R}^2 \longrightarrow S \subseteq \mathbb{R}^3$

$(u,v) \longrightarrow (u, v, 1 - u^2 - v^2)$

$\underline{X}$ dà un'orientazione. È quella voluta?

basta calcolare $N = \dfrac{\underline{X}_u \wedge \underline{X}_v}{\| \quad \|}$

e confrontarlo con quello assegnato

$$\begin{array}{cccc} & i & j & k \\ \underline{X}_u = (& 1, & 0, & -2u), \\ \underline{X}_v = (& 0, & 1, & -2v) \end{array}$$

$\underline{X}_u \wedge \underline{X}_v = (2u, 2v, 1)$

in $P = (0,0,1)$ deve essere $(u,v) = (0,0)$

$\underline{X}_u \wedge \underline{X}_v = (0,0,1)$ OK.

Sulle dispense, esercizio 3.11, pag 275