Giugno 2019, ex. 4 (1)

su $\mathbb{R}^4$ $\quad (x_1, x_2, x_3, x_4)$

$$\omega = x_3 x_4\, dx_1 \wedge dx_2 + x_2 x_4\, dx_1 \wedge dx_3 + x_2 x_3\, dx_1 \wedge dx_4$$

- Calcolare: $d\omega$, $*\omega$, $d(*\omega)$, $\omega \wedge *\omega$
- trovare, se esiste $\eta$ : $\omega = d\eta$

- $$d\omega = d(x_3 x_4) \wedge dx_1 \wedge dx_2 + d(x_2 x_4) \wedge dx_1 \wedge dx_3 + d(x_2 x_3) \wedge dx_1 \wedge dx_4$$

$$= [x_4 dx_3 + x_3 dx_4] \wedge dx_1 \wedge dx_2 + [x_4 dx_2 + x_2 dx_4] \wedge dx_1 \wedge dx_3 + [x_3 dx_2 + x_2 dx_3] \wedge dx_1 \wedge dx_4$$

$$= x_4\, dx_1 \wedge dx_2 \wedge dx_3 + x_3\, dx_1 \wedge dx_2 \wedge dx_4 - x_4\, dx_1 \wedge dx_2 \wedge dx_3 + x_2\, dx_1 \wedge dx_3 \wedge dx_4 - x_3\, dx_1 \wedge dx_2 \wedge dx_4 - x_2\, dx_1 \wedge dx_3 \wedge dx_4$$

$$= 0 \qquad (\omega \text{ è chiusa})$$

②

$$*\omega = *\big(x_3x_4\, dx_1\wedge dx_2 + x_2x_4\, dx_1\wedge dx_3 + x_2x_3\, dx_1\wedge dx_4\big)$$

$$*(dx_1\wedge dx_2) = +\, dx_3\wedge dx_4 \qquad \cdot$$

$$*(dx_1\wedge dx_3) = -\, dx_2\wedge dx_4 \qquad \cdot\cdot$$

$$*(dx_1\wedge dx_4) = +\, dx_2\wedge dx_3 \qquad \cdots$$

$\cdot$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \longrightarrow +$

$\cdot\cdot$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = (2\,3) \longrightarrow -$

$\cdots$ $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = (2\,4\,3) \longrightarrow +$

$$*\omega = \underline{x_3x_4}\, dx_{\underline{\underline{3}}}\wedge dx_{\underline{\underline{4}}} - \underline{x_2x_4}\, dx_{\underline{\underline{2}}}\wedge dx_{\underline{\underline{4}}} + \underline{x_2x_3}\, dx_{\underline{\underline{2}}}\wedge dx_{\underline{\underline{3}}}$$

- $d(*\omega) = 0$ ogni termine ha un $dx_i$ ripetuto.

- $\omega\wedge(*\omega) =$

$$\big[x_3x_4\, dx_1\wedge dx_2 + x_2x_4\, dx_1\wedge dx_3 + x_2x_3\, dx_1\wedge dx_4\big]\wedge$$

$$\big[x_3x_4\, dx_3\wedge dx_4 - x_2x_4\, dx_2\wedge dx_4 + x_2x_3\, dx_2\wedge dx_3\big]$$

$=$

$$(x_3x_4)^2\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$-\,(x_2x_4)^2\, dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4$$
$$+\,(x_2x_3)^2\, dx_1 \wedge dx_4 \wedge dx_2 \wedge dx_3$$

$$= \left[(x_3x_4)^2 + (x_2x_4)^2 + (x_2x_3)^2\right] dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

Osservazione: in $\mathbb{R}^n$ $\quad \omega = \sum a_I\, dx_I$

$\omega$ $k$-forma

$*\omega = (n-k)$ forma $\quad *\omega = \sum (\pm)\, a_I\, dx_J$

dove $I \cup J = \{1, 2, \dots, n\}$

$I+J =$

$$*\, a_I\, dx_I = (-1)^{I+J}\, a_I\, dx_J$$

$= i_1 \dots i_k j_1 \dots j_{n-k}$

$$a_I dx_I \wedge (*\, a_I dx_I) =$$
$$a_I^2 \cdot (-1)^{I+J}\, dx_I \wedge dx_J =$$

$$= \underbrace{(-1)^{I+J} \cdot (-1)^{I+J}}_{+1}\, a_I^2\, dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

quindi in generale:

$$\omega = \sum_I' a_I \, dx_I \qquad k\text{-forma}$$

$$\omega \wedge (*\omega) = \left( \sum_I' a_I^2 \right) dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$$

- trovare $\eta$: $d\eta = \omega$

$\omega$ chiusa $(d\omega = 0)$, $\mathbb{R}^4$ contraibile

$\Longrightarrow$ $\omega$ esatta

trovare $\eta$ = "integrare"

$$\omega = dx_1 \wedge \underbrace{( x_3 x_4 \, dx_2 + x_2 x_4 \, dx_3 + x_2 x_3 \, dx_4 )}_{\varphi}$$

$$\boxed{\omega = dx_1 \wedge \varphi}$$

$$f \quad G \longrightarrow FG - \int \underline{\underline{Fg}}$$

$$\eta = x_1 \varphi - \int 0 = x_1 \varphi$$

$$F = x_1, \quad g = d\varphi = 0$$

$$\eta = x_1 x_3 x \, dx_2 + x_1 x_2 x_4 \, dx_3 + x_1 x_2 x_3 \, dx_4$$

Febbraio 2020 ex. 4

Su $\mathbb{R}^n$

$$\omega = \sum_{i=1}^{n} (-1)^i x_i \, dx_1 \wedge \dots \wedge \hat{dx_i} \wedge \dots \wedge dx_n$$

in $\mathbb{R}^2 = \qquad -x_1 dx_2 + x_2 dx_1$

in $\mathbb{R}^3 = \qquad -x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 - x_3 dx_1 \wedge dx_2$

Calcolare $d\omega$, $*\omega$, $\omega \wedge (*\omega)$

$$d\omega = \sum_{i=1}^{n} (-1)^i \, dx_i \wedge \underbrace{dx_1 \wedge \dots \wedge \hat{dx_i} \wedge \dots \wedge dx_n}$$

$$= \sum_{i=1}^{n} (-1)^i (-1)^{i-1} dx_1 \wedge dx_2 \dots \wedge dx_n$$

$$= -n \left( dx_1 \wedge \dots \wedge dx_n \right)$$

$*\omega$: $\qquad * (dx_1 \wedge dx_2 \wedge \dots \wedge \hat{dx_i} \wedge \dots dx_n) = \pm dx_i$

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & i-1 & i & i+1 & \dots & n-1 & n \\ \underbrace{1 \; 2 & & & i-1} & i+1 & i+2 & \dots & n & i \end{pmatrix}$$

$$\sigma = (i \;\; i+1 \; \dots \; n)$$

$\sigma$ è un ciclo di lunghezza $n-i+1$ ⑥

$\longrightarrow \quad (-1)^{n-i}$

$$*\omega = \sum_{i=1}^{n} (-1)^{i} x_i \, (-1)^{n-i} dx_i$$

$$= \sum_{i=1}^{n} (-1)^{n} x_i \, dx_i$$

$$= \boxed{(-1)^{n} \cdot \sum_{i=1}^{n} x_i \, dx_i}$$

$$\omega \wedge (*\omega) = \left[ \sum_{i=1}^{n} x_i^2 \right] dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$$

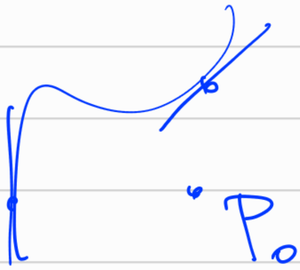
$d\omega$ = $n$ - forma $= (-n) \underbrace{dx_1 \wedge \ldots \wedge dx_n}$

$\omega \wedge (*\omega)$ = $n$ - forma =

$$\boxed{\omega \wedge (*\omega) = \left( -\frac{1}{n} \cdot \sum x_i^2 \right) d\omega}$$

Settembre 2018, ex. 1 (7)

$\sigma : \mathbb{R} \to \mathbb{R}^2$ curva piana, pensan anche



$d(P_0, r) = c$

$r$ = retta tg a $\sigma$

Allora: $\sigma$ = arco di circ. oppure segmento

$r$=costante = $\sigma$

$P_0$ = centro di $\sigma$

distanza (centro, tg) = raggio costante

$$\| a'(t) - P_0 \|^2 = \text{costante}$$

$$\to \left( \| a'(t) - P_0 \|^2 \right)' = 0$$

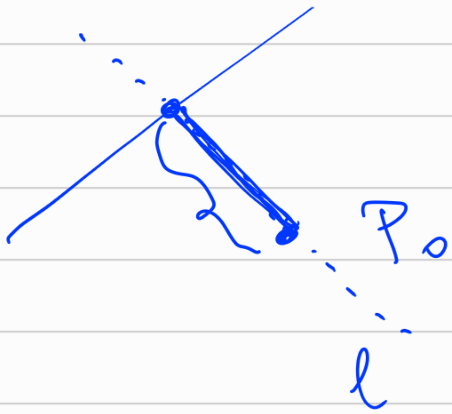
Quindi: $r$ = retta tangente

$d(\text{retta}, \text{punto})^2$ = costante

$\sigma(s)$ $\qquad \sigma'(s)$ = vettore tg

retta tg: $r$: $\sigma(s) + t\sigma'(s)$ = in forma parametrica


d(retta, punto) = ??




$l$: $P_0 + u\,\sigma''(s)$

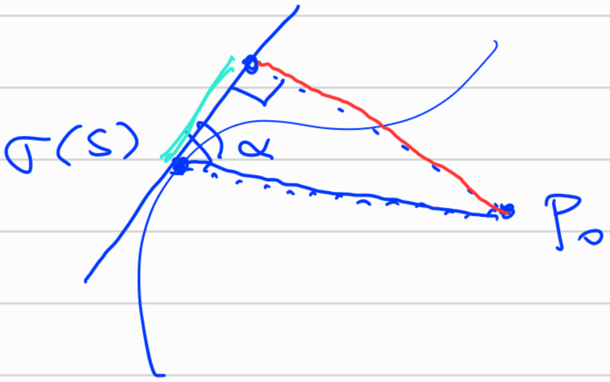
$r$: $\sigma(s) + t\,\sigma'(s)$

$P_0 + u\,\sigma''(s) = \sigma(s) + t\,\sigma'(s)$

$\sigma(s) + \underline{\underline{\theta(s)}}\,\sigma'(s)$ = proiez ort di $P_0$ sulla tg

$\|(\sigma(s) - P_0) + \theta(s)\,\sigma'(s)\|^2$ = costante

$(\quad)\cdot(\quad)$ $\qquad (a \cdot a)' = 2a \cdot a'$




retta tangente

$P_0 = (x_0, y_0)$

$r: \quad \sigma(s) + t\sigma'(s)$

$\sigma(s) - P_0 \quad = \text{vettore}$

trovo $\alpha$ = angolo fra $(\sigma(s) - P_0)$ e $\sigma'(s)$

$\longrightarrow$ trovo la distanza e derivo rispetto a $s$.

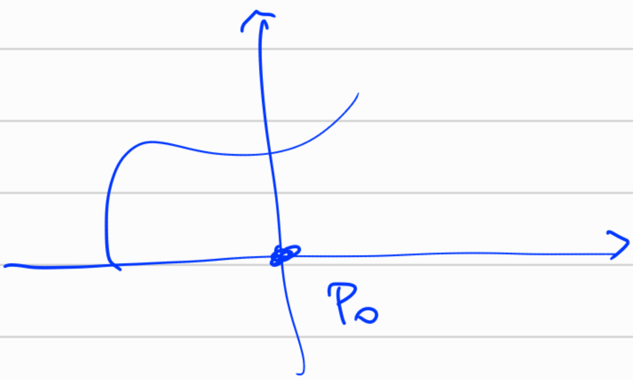
$(\sigma(s) - P_0) \cdot \sigma'(s) = \|\sigma(s) - P_0\| \cdot \cos\alpha$

$\longrightarrow$ si dovrebbe fare.

Altrimenti:

$r: \quad \sigma(s) + t\sigma'(s)$

$l: \quad u \cdot \sigma''(s)$

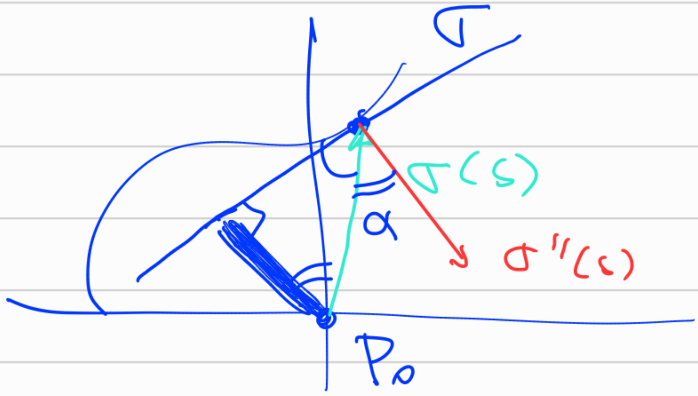



$$\sigma(s) + \underline{t}\,\sigma'(s) = \underline{u}\, k(s)\, \underline{n}(s)$$

$$\sigma'(s) + t'\sigma' + t\sigma'' = u' k(s)\, \underline{n}(s) + u k' \underline{n} + u k \underline{n}'$$

$$\underline{t}(1+t') + tk\underline{n} = u'k\underline{n} + uk'\underline{n} - uk^2\underline{t}$$

$$\underline{t}\,(\underbrace{1+t'+uk^2}_{=0}) + \underline{n}\,(\underbrace{tk-u'k-uk'}_{=0}) = 0$$




$\sigma(s)\cdot\sigma''(s)$

blu = verde $\cdot$ cos ($\alpha$)

$$\sigma(s)\cdot\sigma''(s) = \|\sigma(s)\|\cdot\|\sigma''(s)\|\cos\alpha$$

$$\cancel{\|\sigma(s)\|}\cdot\frac{\sigma(s)\cdot\sigma''(s)}{\cancel{\|\sigma(s)\|}\cdot\|\sigma''(s)\|} =$$

$$= \boxed{\frac{1}{k(s)}\cdot\left(\sigma(s)\cdot\sigma''(s)\right)}$$ biregolare

= costante

$$\left(\sigma(s)\cdot\sigma''(s)\right)' = \sigma'\cdot\sigma'' + \sigma\cdot\sigma''' = 0$$

0

Distanza: $\sigma \cdot \underline{n}$

$$\underbrace{\sigma' \cdot \underline{n}}_{0} + \sigma \cdot \underline{n}' =$$

$$= \sigma \cdot (-k\,\underline{t}) = 0$$

$\tau \equiv 0$ curva piana

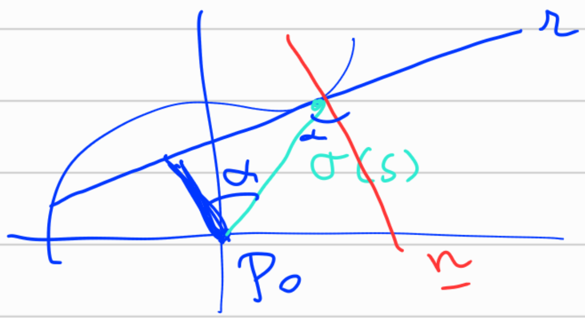
$$\Rightarrow \sigma \perp \underline{t}$$

$$\underline{t} = \sigma' \Rightarrow \sigma \cdot \sigma' = 0$$

$$\Rightarrow (\sigma \cdot \sigma) = \underline{\underline{\text{costante}}}$$

—o—

Dall'inizio $P_0 = 0$




$$d(r, P_0) = \|\sigma(s)\| \cdot \cos\alpha$$

$$\sigma(s) \cdot \underline{n} = \|\sigma(s)\| \cdot \cos\alpha$$

$$\Rightarrow d(r, P_0) = \|\sigma(s)\| \cdot \frac{\sigma(s) \cdot \underline{n}}{\|\sigma(s)\|}$$

$$= \sigma(s) \cdot \underline{n}$$

$$d(r, P_0)'' \equiv 0$$ (12)

$$\sigma' \cdot \underline{n} + \sigma \cdot \underline{n}' =$$

$$= \cancel{\underline{t} \cdot \underline{n}} + \sigma \cdot (-k \underline{t} + \tau \cancel{\underline{b}})$$

$$= -k \cdot (\sigma \cdot \underline{t}) \equiv 0$$

$k \equiv 0 \to$ retta

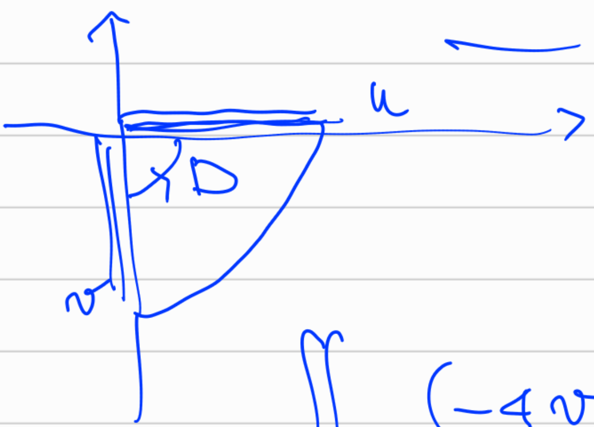
$k \not\equiv 0$ & un intervallo & cui $k(s) \neq 0$

$$\to \sigma \cdot \underline{t} \equiv 0$$

$$\sigma \cdot \sigma' \equiv 0$$

integrando $(\sigma \cdot \sigma) =$ costante

cioè: $\sigma =$ circonferenza di centro $P_0$




$$\iint_D (-4v - 2u)\, du\, dv$$

$$\int_{\partial D} (-2u^2 + 2uv + v^2)\, du + (-2uv + 2v^2)\, dv$$