is not a regular surface. Observe that we cannot conclude this from the fact alone that the "natural" parametrization

$$(x, y) \to (x, y, +\sqrt{x^2 + y^2})$$

is not differentiable; there could be other parametrizations satisfying Def. 1.

To show that this is not the case, we use Prop. 3. If C were a regular surface, it would be, in a neighborhood of $(0, 0, 0) \in C$, the graph of a differentiable function having one of three forms: $y = h(x, z)$, $x = g(y, z)$, $z = f(x, y)$. The two first forms can be discarded by the simple fact that the projections of C over the xz and yz planes are not one-to-one. The last form would have to agree, in a neighborhood of $(0, 0, 0)$, with $z = +\sqrt{x^2 + y^2}$. Since $z = +\sqrt{x^2 + y^2}$ is not differentiable at $(0, 0)$, this is impossible.

Example 6. A parametrization for the torus T of Example 4 can be given by (Fig. 2-9)

$$\mathbf{x}(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u),$$

where $0 < u < 2\pi$, $0 < v < 2\pi$.

Condition 1 of Def. 1 is easily checked, and condition 3 reduces to a straightforward computation, which is left as an exercise. Since we know that T is a regular surface, condition 2 is equivalent, by Prop. 4, to the fact that $\mathbf{x}$ is one-to-one.

To prove that $\mathbf{x}$ is one-to-one, we first observe that $\sin u = z/r$; also, if $\sqrt{x^2 + y^2} \le a$, then $\pi/2 \le u \le 3\pi/2$, and if $\sqrt{x^2 + y^2} \ge a$, then either $0 < u \le \pi/2$ or $3\pi/2 \le u < 2\pi$. Thus, given (x, y, z), this determines u, $0 < u < 2\pi$, uniquely. By knowing u, x, and y we find $\cos v$ and $\sin v$. This determines v uniquely, $0 < v < 2\pi$. Thus, $\mathbf{x}$ is one-to-one.

It is easy to see that the torus can be covered by three such coordinate neighborhoods.

EXERCISES†

1. Show that the cylinder $\{(x, y, z) \in R^3; x^2 + y^2 = 1\}$ is a regular surface, and find parametrizations whose coordinate neighborhoods cover it.
2. Is the set $\{(x, y, z) \in R^3; z = 0 \text{ and } x^2 + y^2 \le 1\}$ a regular surface? Is the set $\{(x, y, z) \in R^3; z = 0, \text{ and } x^2 + y^2 < 1\}$ a regular surface?
3. Show that the two-sheeted cone, with its vertex at the origin, that is, the set $\{(x, y, z) \in R^3; x^2 + y^2 - z^2 = 0\}$, is not a regular surface.

†Those who have omitted the proofs in this section should also omit Exercises 17–19.

4. Let $f(x, y, z) = z^2$. Prove that 0 is not a regular value of f and yet that $f^{-1}(0)$ is a regular surface.

***5.** Let $P = \{(x, y, z) \in R^3; x = y\}$ (a plane) and let $\mathbf{x}: U \subset R^2 \to R^3$ be given by

$$\mathbf{x}(u, v) = (u + v, u + v, uv),$$

where $U = \{(u, v) \in R^2; u > v\}$. Clearly, $\mathbf{x}(U) \subset P$. Is $\mathbf{x}$ a parametrization of P?

6. Give another proof of Prop. 1 by applying Prop. 2 to $h(x, y, z) = f(x, y) - z$.

7. Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

c. Answer the questions of parts a and b for the function $f(x, y, z) = xyz^2$.

8. Let $\mathbf{x}(u, v)$ be as in Def. 1. Verify that $d\mathbf{x}_q: R^2 \to R^3$ is one-to-one if and only if

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

9. Let V be an open set in the xy plane. Show that the set

$$\{(x, y, z) \in R^3; z = 0 \text{ and } (x, y) \in V\}$$

is a regular surface.

10. Let C be a figure "8" in the xy plane and let S be the cylindrical surface over C (Fig. 2-11); that is,

$$S = \{(x, y, z) \in R^3; (x, y) \in C\}.$$

Is the set S a regular surface?

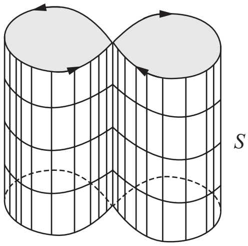

Figure 2-11