The following proposition shows that we can extend the local concepts and properties of differential geometry to regular parametrized surfaces.

PROPOSITION 2. *Let* $\mathbf{x}\colon \mathrm{U} \subset \mathrm{R}^2 \to \mathrm{R}^3$ *be a regular parametrized surface and let* $\mathrm{q} \in \mathrm{U}$. *Then there exists a neighborhood* V *of* q *in* R^2 *such that* $\mathbf{x}(\mathrm{V}) \subset \mathrm{R}^3$ *is a regular surface.*

Proof. This is again a consequence of the inverse function theorem. Write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

By regularity, we can assume that $(\partial(x, y)/\partial(u, v))(q) \neq 0$. Define a map $F\colon U \times R \to R^3$ by

$$F(u, v, t) = (x(u, v), y(u, v), z(u, v) + t), \quad (u, v) \in U, t \in R.$$

Then

$$\det(dF_q) = \frac{\partial(x, y)}{\partial(u, v)}(q) \neq 0.$$

By the inverse function theorem, there exist neighborhoods W_1 of q and W_2 of $F(q)$ such that $F\colon W_1 \to W_2$ is a diffeomorphism. Set $V = W_1 \cap U$ and observe that the restriction $F|V = \mathbf{x}|V$. Thus, $\mathbf{x}(V)$ is diffeomorphic to V, and hence a regular surface. **Q.E.D.**

EXERCISES†

***1.** Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let $A\colon S^2 \to S^2$ be the (*antipodal*) map $A(x, y, z) = (-x, -y, -z)$. Prove that A is a diffeomorphism.

2. Let $S \subset R^3$ be a regular surface and $\pi\colon S \to R^2$ be the map which takes each $p \in S$ into its orthogonal projection over $R^2 = \{(x, y, z) \in R^3; z = 0\}$. Is π differentiable?

3. Show that the paraboloid $z = x^2 + y^2$ is diffeomorphic to a plane.

4. Construct a diffeomorphism between the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and the sphere $x^2 + y^2 + z^2 = 1$.

***5.** Let $S \subset R^3$ be a regular surface, and let $d\colon S \to R$ be given by $d(p) = |p - p_0|$, where $p \in S$, $p_0 \neq R^3$, $p_0 \notin S$; that is, d is the distance from p to a fixed point p_0 not in S. Prove that d is differentiable.

†Those who have omitted the proofs of this section should also omit Exercises 13–16.

6. Prove that the definition of a differentiable map between surfaces does not depend on the parametrizations chosen.

7. Prove that the relation "S_1 is diffeomorphic to S_2" is an equivalence relation in the set of regular surfaces.

***8.** Let $S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$ and $H = \{(x, y, z) \in R^3; x^2 + \mathbf{y}^2 - z^2 = 1\}$. Denote by $N = (0, 0, 1)$ and $S = (0, 0, -1)$ the north and south poles of S^2, respectively, and let $F: S^2 - \{N\} \cup \{S\} \to H$ be defined as follows: For each $p \in S^2 - \{N\} \cup \{S\}$ let the perpendicular from p to the z axis meet $0z$ at q. Consider the half-line l starting at q and containing p. Then $F(p) = l \cap H$ (Fig. 2-20). Prove that F is differentiable.

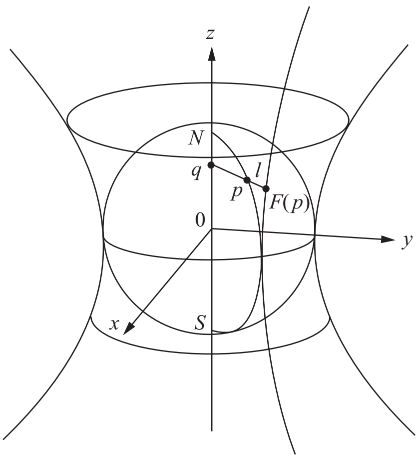

Figure 2-20

9. a. Define the notion of differentiable function on a regular curve. What does one need to prove for the definition to make sense? Do not prove it now. If you have not omitted the proofs in this section, you will be asked to do it in Exercise 15.

b. Show that the map $E: R \to S^1 = \{(x, y) \in R^2; x^2 + y^2 = 1\}$ given by

$$E(t) = (\cos t, \sin t), \quad t \in R,$$

is differentiable (geometrically, E "wraps" R around S^1).

10. Let C be a plane regular curve which lies in one side of a straight line r of the plane and meets r at the points p, q (Fig. 2-21). What conditions should C satisfy to ensure that the rotation of C about r generates an extended (regular) surface of revolution?