is a continuous function on $\mathbf{x}(U)$. Since $\mathbf{x}(U)$ is connected, the sign of f is constant. If $f = -1$, we interchange u and v in the parametrization, and the assertion follows.

Proceeding in this manner with all the coordinate neighborhoods, we have that in the intersection of any two of them, say, $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(\bar{u}, \bar{v})$, the Jacobian

$$\frac{\partial(u, v)}{\partial(\bar{u}, \bar{v})}$$

is certainly positive; otherwise, we would have

$$\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = N(p) = -\frac{\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}}{|\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}|} = -N(p),$$

which is a contradiction. Hence, the given family of coordinate neighborhoods after undergoing certain interchanges of u and v satisfies the conditions of Def. 1, and S is, therefore, orientable. **Q.E.D.**

Remark. As the proof shows, we need only to require the existence of a *continuous* unit vector field on S for S to be orientable. Such a vector field will be automatically differentiable.

Example 3. We shall now describe an example of a nonorientable surface, the so-called *Möbius strip*. This surface is obtained (see Fig. 2-31) by considering the circle S^1 given by $x^2 + y^2 = 4$ and the open segment AB given in the yz plane by $y = 2$, $|z| < 1$. We move the center C of AB along S^1 and turn AB about C in the Cz plane in such a manner that when C has passed through an angle u, AB has rotated by an angle $u/2$. When c completes one trip around the circle, AB returns to its initial position, with its end points inverted.

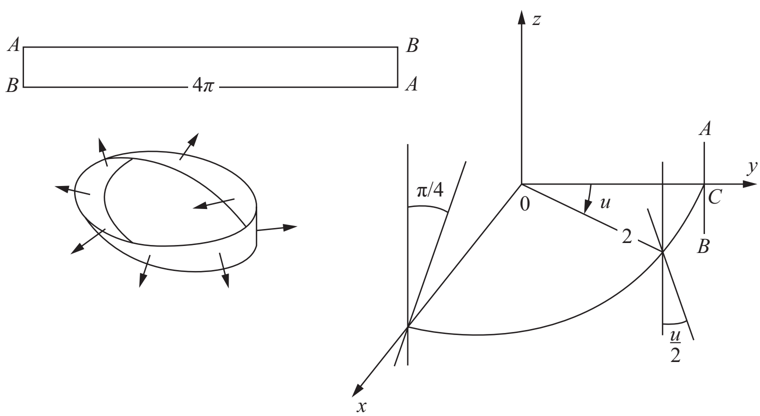

Figure 2-31

From the point of view of differentiability, it is as if we had identified the opposite (vertical) sides of a rectangle giving a twist to the rectangle so that each point of the side AB was identified with its symmetric point (Fig. 2-31).

It is geometrically evident that the Möbius strip M is a regular, nonorieotable surface. In fact, if M were orientable, there would exist a differentiable field $N\colon M \to R^3$ of unit normal vectors. Taking these vectors along the circle $x^2 + y^2 = 4$ we see that after making one trip the vector N returns to its original position as $-N$, which is a contradiction.

We shall now give an analytic proof of the facts mentioned above.

A system of coordinates $\mathbf{x}\colon U \to M$ for the Möbius strip is given by

$$\mathbf{x}(u, v) = \left(\left(2 - v \sin \frac{u}{2}\right) \sin u, \left(2 - v \sin \frac{u}{2}\right) \cos u, v \cos \frac{u}{2}\right),$$

where $0 < u < 2\pi$ and $-1 < v < 1$. The corresponding coordinate neighborhood omits the points of the open interval $u = 0$. Then by taking the origin of the u's at the x axis, we obtain another parametrization $\bar{\mathbf{x}}(\bar{u}, \bar{v})$ given by

$$\begin{aligned} x &= \left\{2 - \bar{v} \sin\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right\} \cos \bar{u}, \\ y &= -\left\{2 - \bar{v} \sin\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right)\right\} \sin \bar{u}, \\ z &= \bar{v} \cos\left(\frac{\pi}{4} + \frac{\bar{u}}{2}\right), \end{aligned}$$

whose coordinate neighborhood omits the interval $u = \pi/2$. These two coordinate neighborhoods cover the Möbius strip and can be used to show that it is a regular surface.

Observe that the intersection of the two coordinate neighborhoods is not connected but consists of two connected components:

$$\begin{aligned} W_1 &= \left\{\mathbf{x}(u, v)\colon \frac{\pi}{2} < u < 2\pi\right\}, \\ W_2 &= \left\{\mathbf{x}(u, v)\colon 0 < u < \frac{\pi}{2}\right\}. \end{aligned}$$

The change of coordinates is given by

$$\left.\begin{aligned} \bar{u} &= u - \frac{\pi}{2} \\ \bar{v} &= v \end{aligned}\right\} \quad \text{in } W_1,$$

and

$$\left.\begin{aligned} \bar{u} &= \frac{3\pi}{2} + u \\ \bar{v} &= -v \end{aligned}\right\} \quad \text{in } W_2.$$

It follows that

$$\frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} = 1 > 0 \quad \text{in } W_1$$

and that

$$\frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} = -1 < 0 \quad \text{in } W_2.$$

To show that the Möbius strip is nonorientable, we suppose that it is possible to define a differentiable field of unit normal vectors $N: M \to R^3$. Interchanging u and v if necessary, we can assume that

$$N(p) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$$

for any p in the coordinate neighborhood of $\mathbf{x}(u, v)$. Analogously, we may assume that

$$N(p) = \frac{\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}}{|\bar{\mathbf{x}}_{\bar{u}} \wedge \bar{\mathbf{x}}_{\bar{v}}|}$$

at all points of the coordinate neighborhood of $\bar{\mathbf{x}}(\bar{u}, \bar{v})$. However, the Jacobian of the change of coordinates must be -1 in either W_1 or W_2 (depending on what changes of the type $u \to v$, $\bar{u} \to \bar{v}$ has to be made). If p is a point of that component of the intersection, then $N(p) = -N(p)$, which is a contradiction.

We have already seen that a surface which is the graph of a differentiable function is orientable. We shall now show that a surface which is the inverse image of a regular value of a differentiable function is also orientable. This is one of the reasons it is relatively difficult to construct examples of nonorientable, regular surfaces in R^3.

PROPOSITION 2. *If a regular surface is given by* $S = \{(x, y, z) \in R^3; f(x, y, z) = a\}$, *where* $f: U \subset R^3 \to R$ *is differentiable and* a *is a regular value of* f, *then* S *is orientable.*

Proof. Given a point $(x_0, y_0, z_0) = p \in S$, consider the parametrized curve $(x(t), y(t), z(t))$, $t \in I$, on S passing through p for $t = t_0$. Since the curve is on S, we have

$$f(x(t), y(t), z(t)) = a$$

for all $t \in I$. By differentiating both sides of this expression with respect to t, we see that at $t = t_0$

$$f_x(p)\left(\frac{dx}{dt}\right)_{t_0} + f_y(p)\left(\frac{dy}{dt}\right)_{t_0} + f_z(p)\left(\frac{dz}{dt}\right)_{t_0} = 0.$$