This agrees with the value found by elementary calculus, say, by using the theorem of Pappus for the area of surfaces of revolution (cf. Exercise 11).

## EXERCISES

**1.** Compute the first fundamental forms of the following parametrized surfaces where they are regular:

**a.** $\mathbf{x}(u, v) = (a \sin u \cos v, b \sin u \sin v, c \cos u)$; ellipsoid.

**b.** $\mathbf{x}(u, v) = (au \cos v, bu \sin v, u^2)$; elliptic paraboloid.

**c.** $\mathbf{x}(u, v) = (au \cosh v, bu \sinh v, u^2)$; hyperbolic paraboloid.

**d.** $\mathbf{x}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$; hyperboloid of two sheets.

**2.** Let $\mathbf{x}(\varphi, \theta) = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$ be a parametrization of the unit sphere $S^2$. Let $P$ be the plane $x = z \operatorname{cotan} \alpha$, $0 < \alpha < \pi$, and $\beta$ be the acute angle which the curve $P \cap S^2$ makes with the semimeridian $\varphi = \varphi_0$. Compute $\cos\beta$.

**3.** Obtain the first fundamental form of the sphere in the parametrization given by stereographic projection (cf. Exercise 16, Sec. 2-2).

**4.** Given the parametrized surface

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, \log \cos v + u), \qquad -\frac{\pi}{2} < v < \frac{\pi}{2},$$

show that the two curves $\mathbf{x}(u, v_1)$, $\mathbf{x}(u, v_2)$ determine segments of equal lengths on all curves $\mathbf{x}(u, \text{const.})$.

**5.** Show that the area $A$ of a bounded region $R$ of the surface $z = f(x, y)$ is

$$A = \iint_Q \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy,$$

where $Q$ is the normal projection of $R$ onto the $xy$ plane.

**6.** Show that

$$\mathbf{x}(u, v) = (u \sin\alpha \cos v, u \sin\alpha \sin v, u \cos\alpha)$$
$$0 < u < \infty, \quad 0 < v < 2\pi, \quad \alpha = \text{const.},$$

is a parametrization of the cone with $2\alpha$ as the angle of the vertex. In the corresponding coordinate neighborhood, prove that the curve

$$\mathbf{x}(c \exp(v \sin\alpha \operatorname{cotan} \beta), v), \qquad c = \text{const.}, \beta = \text{const.},$$

intersects the generators of the cone ($v = \text{const.}$) under the constant angle $\beta$.