$$A' = \iint_R |N_u \wedge N_v|\, du\, dv.$$

Using Eq. (1), the definition of K, and the above convention, we can write

$$A' = \iint_R K|\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv. \tag{12}$$

Going to the limit and denoting also by R the area of the region R, we obtain

$$\lim_{A\to 0} \frac{A'}{A} = \lim_{R\to 0} \frac{A'/R}{A/R} = \frac{\lim\limits_{R\to 0}(1/R) \iint_R K|\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv}{\lim\limits_{R\to 0}(1/R) \iint_R |\mathbf{x}_u \wedge \mathbf{x}_v|\, du\, dv}$$
$$= \frac{K|\mathbf{x}_u \wedge \mathbf{x}_v|}{|\mathbf{x}_u \wedge \mathbf{x}_v|} = K$$

(notice that we have used the mean value theorem for double integrals), and this proves the proposition. **Q.E.D.**

Remark. Comparing the proposition with the expression of the curvature

$$k = \lim_{s\to 0} \frac{\sigma}{s}$$

of a plane curve C at p (here s is the arc length of a small segment of C containing p, and σ is the arc length of its image in the indicatrix of tangents; cf. Exercise 3 of Sec. 1-5), we see that the Gaussian curvature K is the analogue, for surfaces, of the curvature k of plane curves.

EXERCISES

1. Show that at the origin $(0, 0, 0)$ of the hyperboloid $z = axy$ we have $K = -a^2$ and $H = 0$.

***2.** Determine the asymptotic curves and the lines of curvature of the helicoid $x = v \cos u$, $y = v \sin u$, $z = cu$, and show that its mean curvature is zero.

***3.** Determine the asymptotic curves of the catenoid

$$\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v).$$

4. Determine the asymptotic curves and the lines of curvature of $z = xy$.

5. Consider the parametrized surface (Enneper's surface)

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

and show that

a. The coefficients of the first fundamental form are

$$E = G = (1 + u^2 + v^2)^2, \quad F = 0.$$

b. The coefficients of the second fundamental form are

$$e = 2, \quad g = -2, \quad f = 0.$$

c. The principal curvatures are

$$k_1 = \frac{2}{(1 + u^2 + v^2)^2}, \quad k_2 = -\frac{2}{(1 + u^2 + v^2)^2}.$$

d. The lines of curvature are the coordinate curves.

e. The asymptotic curves are $u + v = \text{const.}$, $u - v = \text{const}$.

6. (*A Surface with* $K \equiv -1$; *the Pseudosphere.*)

***a.** Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the *tractrix*; see Fig. 1-9).

b. Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the *pseudosphere*; see Fig. 3-22) is regular and find out a parametrization in a neighborhood of a regular point.

c. Show that the Gaussian curvature of any regular point of the pseudosphere is -1.

7. (*Surfaces of Revolution with Constant Curvature.*) $(\varphi(v)\cos u, \varphi(v)\sin u, \psi(v))$, $\varphi \neq 0$ is given as a surface of revolution with constant Gaussian curvature K. To determine the functions φ and ψ, choose the parameter v in such a way that $(\varphi')^2 + (\psi')^2 = 1$ (geometrically, this means that v is the arc length of the generating curve $(\varphi(v), \psi(v))$). Show that

a. φ satisfies $\varphi'' + K\varphi = 0$ and ψ is given by $\psi = \int \sqrt{1 - (\varphi')^2}\, dv$; thus, $0 < u < 2\pi$, and the domain of v is such that the last integral makes sense.

b. All surfaces of revolution with constant curvature $K = 1$ which intersect perpendicularly the plane xOy are given by

$$\varphi(v) = C\cos v, \quad \psi(v) = \int_0^v \sqrt{1 - C^2 \sin^2 v}\, dv,$$

where C is a constant ($C = \varphi(0)$). Determine the domain of v and draw a rough sketch of the profile of the surface in the xz plane for

b. The length of the segment of a tangent line to a curve $v = \text{const.}$, determined by its point of tangency and the z axis, is constantly equal to 1. Conclude that the curves $v = \text{const.}$ are tractrices (cf. Exercise 6).

13. Let $F\colon R^3 \to R^3$ be the map (a similarity) defined by $F(p) = cp$, $p \in R^3$, c a positive constant. Let $S \subset R^3$ be a regular surface and set $F(S) = \bar{S}$. Show that $\bar{S}$ is a regular surface, and find formulas relating the Gaussian and mean curvatures, K and H, of S with the Gaussian and mean curvatures, $\bar{K}$ and $\bar{H}$, of $\bar{S}$.

14. Consider the surface obtained by rotating the curve $y = x^3$, $-1 < x < 1$, about the line $x = 1$. Show that the points obtained by rotation of the origin $(0, 0)$ of the curve are planar points of the surface.

***15.** Give an example of a surface which has an isolated parabolic point p (that is, no other parabolic point is contained in some neighborhood of p).

***16.** Show that a surface which is compact (i.e., it is bounded and closed in R^3) has an elliptic point.

17. Define Gaussian curvature for a nonorientable surface. Can you define mean curvature for a nonorientable surface?

18. Show that the Möbius strip of Fig. 3-1 can be parametrized by

$$\mathbf{x}(u, v) = \left(\left(2 - v \sin \frac{u}{2}\right) \sin u,\ \left(2 - v \sin \frac{u}{2}\right) \cos u,\ v \cos \frac{u}{2}\right)$$

and that its Gaussian curvature is

$$K = -\frac{1}{\{\frac{1}{4}v^2 + (2 - v\sin(u/2))^2\}^2}.$$

***19.** Obtain the asymptotic curves of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$.

20. Determine the umbilical points of the elipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

***21.** Let S be a surface with orientation N. Let $V \subset S$ be an open set in S and let $f\colon V \subset S \to R$ be any nowhere-zero differentiable function in V. Let v_1 and v_2 be two differentiable (tangent) vector fields in V such that at each point of V, v_1 and v_2 are orthonormal and $v_1 \wedge v_2 = N$.