Curves

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These notes summarize the key points in the first chapter of *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo. I wrote them to assure that the terminology and notation in my lecture agrees with that text. *All page references in these notes are to the Do Carmo text.*

1. Definition. A **parameterized smooth curve** is a map $\alpha : I \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$ is an interval. The set theoretic image

$$C = \alpha(I) := \{\alpha(t) : t \in I\}$$

is called the **trace** of α and α is called a **parameterization** of C. See do Carmo page 2.

2. Remark. For do Carmo the words *differentiable* and *smooth* are synonymous. I prefer the word *smooth*. The adjective *differentiable* is often omitted by do Carmo.

3. Remark. On page 2 do Carmo says that the interval I should be open but on page 30 he extends the notion of smoothness to closed intervals. A function defined on a closed interval $[a, b]$ is said to be **smooth** iff it extends to an open interval containing $[a, b]$. This means that the derivatives of the function are defined at the end points a and b.

4. Definition. A **reparameterization** of $\alpha : I \to \mathbb{R}^n$ is a smooth map $\beta : J \to \mathbb{R}^n$ of form $\beta = \alpha \circ \sigma$ where $\sigma : J \to I$ is a diffeomorphism. That σ is a diffeomorphism means σ is one-to-one and onto (so there is an inverse map $\sigma^{-1} : J \to I$) and that $\sigma'(t) \neq 0$ for $t \in I$ (so that the map σ^{-1} is also smooth).

5. Remark. If β is a reparameterization of α, then the maps α and β have the same trace C. The idea of the definition is that we should think of α and β as different ways of describing the same curve C. However do Carmo avoids giving a precise definition of an (unparameterized) curve.

6. Example. The circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is the trace of the parameterized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\alpha(\theta) = \bigl(\cos\theta, \sin\theta\bigr) = \bigl(\cos(\theta + 2\pi), \sin(\theta + 2\pi)\bigr).$$

Define a map $\beta : \mathbb{R} \to \mathbb{R}^2$ by

$$\beta(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

This map is a reparameterization of the restriction of α to the open interval $(-\pi, \pi)$ as follows:

$$\begin{aligned} \big(\cos(2\varphi), \sin(2\varphi)\big) &= \left(\frac{\cos^2\varphi - \sin^2\varphi}{\cos^2\varphi + \sin^2\varphi}, \frac{2\sin\varphi\cos\varphi}{\cos^2\varphi + \sin^2\varphi} \right) \\ &= \left(\frac{1-\tan^2\varphi}{1+\tan^2\varphi}, \frac{2\tan\varphi}{1+\tan^2\varphi} \right). \end{aligned}$$

Take $2\varphi = \theta$, $t = \tan\varphi = \tan(\theta/2)$, and we get $\alpha = \beta \circ \sigma$ where $\sigma : (-\pi, \pi) \to \mathbb{R}$ is defined by $\sigma(\theta) = \tan(\theta/2)$. The common trace of (the restriction of) α and the map β is the punctured circle $C \setminus (-1, 0)$. (This particular reparameterization is called the **Weierstrass substitution** or **half angle substitution**. It is one of the main techniques used to evaluate integrals in calculus.)

7. Definition. Let $\alpha : I \to \mathbb{R}^n$ be a smooth parameterized curve. The derivative $\alpha'(t)$ is called **velocity vector** at t. The map α is called **regular** iff its velocity vector never vanishes. The map α is said to be **parameterized by arc length** iff its tangent vector always has length one.

8. Theorem. *A smooth regular parameterized curve α has a reparameterization by arc length, i.e. there is a reparameterization $\beta : J \to \mathbb{R}^n$ of α such that $|\beta'(s)| = 1$ for $s \in J$.*

Proof: This is the content of Remark 2 in do Carmo page 21. The reparametrization is defined by $\beta = \alpha \circ \sigma$ where σ is a solution of the differential equation

$$\sigma'(s) = \frac{1}{|\alpha'(\sigma(s))|}.$$

By the chain rule $\beta'(s) = \alpha'(\sigma(s))\sigma'(s)$ so $|\beta'(s)| = 1$. $\square$

9. Remark. The arc length

$$\ell(C) = \int_a^b |\alpha'(t)|\, dt$$

of the trace C of a regular parameterized curve $\alpha : [a, b] \to \mathbb{R}^n$ is independent of the parameterization α used to define it. This is an easy consequence of the formula for changing variables in a definite integral: if $\sigma : [a, b] \to [c, d]$ is a diffeomorphism, then

$$\int_a^b |\alpha'(t)|\, dt = \int_c^d |\alpha'(\sigma(r))|\, |\sigma'(r)|\, dr.$$

(The change of variables is $t = \sigma(r)$ so $dt = \sigma'(r)\, dr$.) When α is parameterized by arc length, $\ell(C) = |b - a|$.

10. The reparameterization in Theorem 8 is unique in the following sense: If $\beta_1 : J_1 \to \mathbb{R}^n$ and $\beta_2 : J_2 \to \mathbb{R}^n$ are two reparameterizations of the same map α then $\beta_2 = \beta_1 \circ \sigma$ where $\sigma : J_2 \to J_1$ has one of the two forms $\sigma(s) = s + c$ or $\sigma(s) = -s + c$. (This is because $|\sigma'(s)| = 1$.) On page 6 do Carmo says that when $\sigma(s) = -s + c$ the two curves β_1 and β_2 are said to **differ by a change of orientation**.

This use of the word *orientation* can be viewed as a special case of the definition of *orientation of a vector space* that do Carmo gives on pages 11 and 12. For a regular curve α the one dimensional vector space $\mathbb{R}\alpha'(t) \subseteq \mathbb{R}^n$ is called the **tangent space** to the curve at the point $\alpha(t)$. The velocity vector $\alpha'(t)$ is a basis for this space. Changing the orientation of the curve changes the sign of the velocity vector $\alpha'(t)$ and thus reverses the orientation of the tangent space.

11. Remark. Note the distinction between the *tangent space* and the *tangent line*. The **tangent line** is the line containing the points $\alpha(t)$ and $\alpha(t) + \alpha'(t)$. (See do Carmo page 5.) This line need not pass through the origin of $\mathbb{R}^n$ and thus is not a vector subspace of the vector space $\mathbb{R}^n$. This illustrates the difference between points and vectors.

12. Remark. The two orientations of $\mathbb{R}^3$ correspond to the thumb, forefinger, and middle finger of the right and left hands. (Recall the *right hand rule* from calculus.) The two orientations of $\mathbb{R}^2$ correspond to *clockwise* and *counter clockwise*. The two orientations of $\mathbb{R} = \mathbb{R}^1$ correspond to the two directions *increasing* and *decreasing*.

13. Definition. A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** iff it preserves distance i.e. iff it satisfies

$$|\Phi(p) - \Phi(q)| = |p - q|$$

for $p, q \in \mathbb{R}^n$. A map $\mathbb{R}^n \to \mathbb{R}^n$ is called a **translation** iff there is a vector $\mathbf{c} \in \mathbb{R}^n$ such that the map sends the point $p \in \mathbb{R}^n$ to the point $p + \mathbf{c}$. A linear transformation $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is called **orthogonal** iff it satisfies $(\rho\mathbf{u})\cdot(\rho\mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. A **rigid motion** of $\mathbb{R}^n$ is an isometry Φ such that the corresponding orthogonal linear transformation ρ preserves orientation, i.e. has positive determinant.

14. Theorem. *A map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if and only if it is the composition of a translation and an orthogonal linear transformation.*

Proof: For *if* see do Carmo page 23 Exercise 6 and do Carmo page 228 Exercise 7. The converse is not very difficult but is not needed in the rest of these notes so the proof is omitted. □

15. Theorem. *Let $\alpha : [a, b] \to \mathbb{R}^n$ be smooth, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, and $\beta = \Phi \circ \alpha$:. Then the curves α and β have the same arc length. If α is parameterized by arc length so is β.*

Proof: This is because Φ preserves the length of vectors. The first part also follows from Exercise 8 on page 10 of do Carmo. □

16. Definition. Let $\alpha : I \to \mathbb{R}^n$ be parameterized by arc length. Then the **unit tangent vector** is the vector valued function $\mathbf{t} : I \to \mathbb{R}^n$ defined by

$$\mathbf{t}(s) = \alpha'(s) = \frac{d}{ds}\alpha(s),$$

the **curvature vector** is the vector valued function $I \to \mathbb{R}^n$

$$\alpha''(s) = \frac{d}{ds}\mathbf{t}(s) = \frac{d^2}{ds^2}\alpha(s),$$

and the **curvature** is the length κ of the curvature vector, i.e.

$$\kappa(s) = |\mathbf{t}'(s)| = |\alpha''(s)|.$$

The **unit normal vector** is the normalized curvature vector

$$\mathbf{n} = \frac{\mathbf{t}'}{|\mathbf{t}'|}.$$

(The vector $\mathbf{n}$ is defined only where the curvature κ is not zero.) The **binormal vector** is the vector product

$$\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$$

of the unit tangent vector $\mathbf{t}$ and the unit normal vector $\mathbf{n}$. (The binormal vector is defined only when $n = 3$.)

17. Theorem. *Let $\alpha : I \to \mathbb{R}^n$ be parametrized by arc length, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry, and $\beta = \Phi \circ \alpha : I \to \mathbb{R}^n$. Then β is also parametrized by arc length and α and β have the same curvature. If $n = 3$ and Φ is a rigid motion they have the same torsion.*

Proof: Exercise 6 page 23 of do Carmo. □

18. Standing Assumption. *Henceforth we assume that $\alpha : I \to \mathbb{R}^3$ is a regular curve parameterized by arc length.*

19. Theorem. *Then the vectors* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ *are orthonormal, i.e.*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1, \qquad \mathbf{t} \cdot \mathbf{n} = \mathbf{t} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{b} = 0.$$

The ordered orthonormal basis $\mathbf{t}, \mathbf{n}, \mathbf{b}$ *is called the* **Frenet trihedron**.

Proof: (See do Carmo pages 18-19.) The equations $|\mathbf{t}| = |\mathbf{n}| = 1$ hold by definition. Since $|\mathbf{t}|^2 = \mathbf{t} \cdot \mathbf{t}$ is constant we get

$$0 = \frac{d}{ds}|\mathbf{t}|^2 = \frac{d}{ds}\mathbf{t} \cdot \mathbf{t} = 2\,\mathbf{t} \cdot \mathbf{t}' = 2\kappa\,\mathbf{t} \cdot \mathbf{n}$$

so $\mathbf{t} \cdot \mathbf{n} = 0$. Now $\mathbf{b}$ is the vector product of two orthogonal unit vectors $\mathbf{t}$ and $\mathbf{n}$ so it is itself a unit vector and is orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. □

20. Corollary. *The derivative* $\mathbf{b}'$ *of the binormal vector* $\mathbf{b}$ *is parallel to the unit normal vector* $\mathbf{n}$, *i.e. there is a real valued function* τ *such that*

$$\mathbf{b}' = \tau\mathbf{n}, \qquad \tau = \mathbf{b}' \cdot \mathbf{n}.$$

The function τ *is called the* **torsion**.

Proof: Since $\mathbf{t}' \wedge \mathbf{n} = \kappa\,\mathbf{n} \wedge \mathbf{n} = 0$ we have

$$\mathbf{b}' = (\mathbf{t} \wedge \mathbf{n})' = \mathbf{t}' \wedge \mathbf{n} + \mathbf{t} \wedge \mathbf{n}' = \mathbf{t} \wedge \mathbf{n}'$$

so $\mathbf{b}' \cdot \mathbf{t} = \mathbf{b}' \cdot \mathbf{n}' = 0$. □

21. Frenet Formulas. *The Frenet trihedron satisfies the differential equations*

$$\boxed{\mathbf{t}' = \kappa\mathbf{n}, \qquad \mathbf{n}' = -\kappa\mathbf{t} - \tau\mathbf{b}, \qquad \mathbf{b}' = \tau\mathbf{n}.}$$

Proof: The first and last formulas hold by definition. For the middle formula differentiate the identities $\mathbf{n} \cdot \mathbf{t} = \mathbf{n} \cdot \mathbf{b} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$ to get

$$\begin{aligned}
0 &= \mathbf{n}' \cdot \mathbf{t} + \mathbf{n} \cdot \mathbf{t}' = \mathbf{n}' \cdot \mathbf{t} + \kappa\,\mathbf{n} \cdot \mathbf{n} = \mathbf{n}' \cdot \mathbf{t} + \kappa \\
0 &= \mathbf{n}' \cdot \mathbf{b} + \mathbf{n} \cdot \mathbf{b}' = \mathbf{n}' \cdot \mathbf{t} + \tau\,\mathbf{n} \cdot \mathbf{n} = \mathbf{n}' \cdot \mathbf{b} + \tau \\
0 &= 2\mathbf{n}' \cdot \mathbf{n}
\end{aligned}$$

Since the Frenet trihedron is orthonormal

$$\mathbf{n}' = (\mathbf{n}' \cdot \mathbf{t})\mathbf{t} + (\mathbf{n}' \cdot \mathbf{n})\mathbf{n} + (\mathbf{n}' \cdot \mathbf{b})\mathbf{b}.$$

This proves the middle Frenet formula. □

22. Remark. The Frenet formulas may be written in matrix form as

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}$$

The coefficient matrix is skew symmetric. This is no coincidence. The two triples

$$\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s), \qquad \mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)$$

are both bases for the vector space $\mathbb{R}^3$ so there is a unique change of basis matrix $U(s)$ satisfying

$$\begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}(s) \end{pmatrix} = U(s) \begin{pmatrix} \mathbf{t}(s_0) \\ \mathbf{n}(s_0) \\ \mathbf{b}(s_0 \end{pmatrix}.$$

Since the two bases are both orthonormal, the matrix $U(s)$ is orthogonal. Differentiating with respect to s and evaluating at $s = s_0$ gives the Frenet formula (in matrix form) evaluated at $s = s_0$. But $U(s_0)$ is the identity matrix and $U(s)^* = U(s)^{-1}$ so $U^*(s)U(s)$ is the identity matrix so differentiating at s and evaluating at s_0 gives

$$U'(s_0)^* + U'(s_0) = 0.$$ □

23. Theorem. *Reversing the orientation of α leaves the curvature κ and the torsion τ unchanged, i.e. if $\beta(s) = \alpha(-s)$ the curves α and β have the same curvature and torsion at $s = 0$.*

Proof: By definition the curvature κ is nonnegative, the normal vector is only defined at points where the curvature κ is not zero, reversing the orientation of α reverses the sign of the unit tangent vector $\mathbf{t}$ and leaves the sign of the curvature vector unchanged. Reversing the orientation of α reverses the sign of $\mathbf{t}$, preserves the sign of $\mathbf{n}$, and therefore reverses the sign of $\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$. But reversing the orientation of $\mathbf{b}$ reverses the sign of $\mathbf{b}'$ so reversing the orientation of α preserves the sign of $\mathbf{b}'$ and hence (by the Frenet formula $\mathbf{b}' = \tau\mathbf{n}$) preserves the sign of τ. □

24. Fundamental Theorem. *Let $\kappa, \tau : I \to \mathbb{R}$ be smooth functions defined on an interval I. Assume $\kappa > 0$. Then*

(Existence.) There is a curve $\alpha : I \to \mathbb{R}^3$ parameterized by arc length with curvature κ and torsion τ.

(Uniqueness.) If $\alpha, \beta : I \to \mathbb{R}^3$ are two curves paramaterized by arc length both having curvature κ and torsion τ, then there is a rigid motion $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\beta = \Phi \circ \alpha$.

Proof: See do Carmo page 309. □

25. Corollary. The curvature and torsion of the helix $\alpha(\theta) = (a\cos\theta, a\sin\theta, b\theta)$ are both constant so for any two points p and q on the helix there is a rigid motion carrying p to q and mapping the helix to itself.

26. Gauss curvature. In the case of a plane curve ($n = 2$) it is possible to choose a normal vector even when the curvature is zero. In this case since $\mathbf{t}$ and $\mathbf{n}$ are orthogonal unit vectors we can define $\mathbf{n}$ by rotating $\mathbf{t}$ clockwise through 90 degrees:

$$\mathbf{t} = (\xi, \eta), \qquad \mathbf{n} = (\eta, -\xi).$$

With this definition both $\mathbf{t}$ and $\mathbf{n}$ change sign when the orientation is reversed so to maintain the equation $\mathbf{t}' = \kappa\mathbf{n}$ it is necessary to allow κ to be negative. For a plane curve $\alpha : I \to \mathbb{R}^2$ parameterized by arc length we can view the unit normal vector as a map to the unit circle and define an angle $\theta = \theta(s)$ by the formula

$$\mathbf{n}(s) = (\cos\theta(s), \sin\theta(s)).$$

We then define the **signed curvature** by the formula

$$\kappa = \frac{d\theta}{ds}.$$

The signed curvature κ for a plane curve $C \subseteq \mathbb{R}^2$ is analogous to the Gauss curvature K of a surface $S \subseteq \mathbb{R}^3$. (See do Carmo pages 146, 155, 167.) Note that when $\alpha(s) = (\cos s, \sin s)$ is the counter clockwise parameterization of the unit circle in $\mathbb{R}^2$, the vector $\mathbf{n}$ defined by rotation of $\mathbf{t}$ as above is the outward normal (=radius vector) to the circle and the curvature κ is identically one. Thus the curvature compares the curve α to the unit circle.

27. Setup for local canonical form. Assume that $\alpha : I \to \mathbb{R}^3$ has positive curvature and $s_0 \in I$. The Taylor expansion

$$\alpha(s) = \alpha(s_0) + (s - s_0)\alpha'(s_0) + \frac{(s - s_0)^2}{2}\alpha''(s_0) + \frac{(s - s_0)^3}{6}\alpha'''(s_0) + \cdots$$

tells us what the trace C of α looks like near the point $\alpha(s_0) \in \mathbb{C}$. Because any reparameterization of C has the same trace we assume that α is parameterized by arc length. Because the reparameterization defined by $\sigma(s) = s - s_0$ is also a parameterization by arc length, we assume that $s_0 = 0$. Because the arc length, curvature, and torsion are invariant under rigid motions, we assume that

$$\alpha(0) = (0,0,0), \qquad \mathbf{t}(0) = (1,0,0), \qquad \mathbf{n}(0) = (0,1,0), \qquad \mathbf{b}(0) = (0,0,1).$$

28. Local Canonical Form. *In the notation of Setup 27 above, the Taylor expansion of $\alpha(s) = (x(s), y(s), z(s))$ is*

$$\begin{aligned} x(s) &= s - \frac{\kappa(0)s^3}{6} + R_x \\ y(s) &= \frac{\kappa(0)s^2}{2} - \frac{\kappa'(0)s^3}{6} + R_y \\ z(s) &= -\frac{\kappa(0)\tau(0)s^3}{6} + R_z \end{aligned}$$

where $R_x, R_y, R_z = o(s^3)$.

Proof: There is no constant term in these formulas because $\alpha(0) = 0$. By definition

$$\alpha' = \mathbf{t}, \qquad \alpha'' = \kappa\mathbf{n}.$$

Differentiating once more gives

$$\alpha''' = \kappa'\mathbf{n} + \kappa\mathbf{n}' = \kappa'\mathbf{n} + \kappa(-\kappa\mathbf{t} - \tau\mathbf{b})$$

by the second Frenet formula. Now evaluate at $s = s_0 = 0$. $\square$

29. Application. Recall (Remark 11 above and do Carmo page 5) that the **tangent line** to the trace C of a regular curve α at a point $p_0 = \alpha(s_0) \in C$ is the line containing the two points p_0 and $p_0 + \mathbf{t}_0$ where $\mathbf{t}_0 = \mathbf{t}(s_0)$. The **osculating plane** to C at p_0 is the plane containing the three points p_0, $p_0 + \mathbf{t}_0$, $p_0 + \mathbf{n}_0$ where $\mathbf{n}_0 = \mathbf{n}(s_0)$. (See do Carmo pages 17, 29, 30. The definition assumes that the curvature $\kappa(s_0)$ at p_0 is positive.) Let $p_1 = \alpha(s_1)$ and $p_2 = \alpha(s_2)$ be two other points on C distinct from p_0 and each other. Then as $s_1 \to s_0$ the limit of the line through p_0 and p_1 is the tangent line at p_0 and the limit as $s_1, s_2 \to s_0$ of the plane through p_0, p_1, and p_2 is the osculating plane.

Surfaces

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These notes summarize the key points in the second chapter of *Differential Geometry of Curves and Surfaces* by Manfredo P. do Carmo. I wrote them to assure that the terminology and notation in my lecture agrees with that text.

1. Notation. Throughout $\mathbf{x} : U \to \mathbb{R}^3$ is a smooth[1] map defined of an open set $U \subseteq \mathbb{R}^2$ in the plane. Usually a typical point of U denoted by $q = (u, v)$ and the components of the map $\mathbf{x}$ are denoted

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

The **differential** of this map at $q \in \mathbb{R}^2$ is the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ represented by the matrix of partial derivatives

$$d\mathbf{x}_q = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} \end{pmatrix}$$

evaluated at the point $q = (u, v)$. See do Carmo page 54. On page 84 he introduces the notations

$$\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u} = \begin{pmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial u} \\ \dfrac{\partial z}{\partial u} \end{pmatrix}, \qquad \mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v} = \begin{pmatrix} \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial v} \\ \dfrac{\partial z}{\partial v} \end{pmatrix}.$$

for the columns of $d\mathbf{x}_q$. Note the inconsistency of notation: in the expression $d\mathbf{x}_q$ the subscript q indicates where the partial derivatives are to be evaluated while in the expressions $\mathbf{x}_u$ and $\mathbf{x}_v$ the subscript indicates which partial derivative is being computed.

[1] For do Carmo the terms *smooth*, *differentiable* and *infinitely differentiable* are synonymous. I prefer the term *smooth*.

2. Definition. A **parameterized surface** is a map $\mathbf{x} : U \to \mathbb{R}^3$ as above. The image $\mathbf{x}(U) \subseteq \mathbb{R}^3$ is called the **trace** and the surface is called **regular** iff the differential $d\mathbf{x}_q$ is one-to-one for all $q \in U$. (See do Carmo page 78.)

3. Remarks. The definition is analogous to the definition of *regular parameterized curve* $\alpha : I \to \mathbb{R}^3$ given on pages 2 and 6 of do Carmo. The condition that $d\mathbf{x}_q$ be one-to-one holds if and only if $\mathbf{x}_u \wedge \mathbf{x}_v \neq 0$ and this is the analog of the regularity condition that $\alpha' \neq 0$. As for curves the real object of study is the trace. The following definitions restrict the trace and also enable us to define surfaces independently from any particular parameterization.

4. Definition. A subset $S \subseteq \mathbb{R}^3$ of $\mathbb{R}^3$ is called a **regular surface** iff for every point $p_0 \in S$ there is an open subset $V \subseteq \mathbb{R}^3$ and a regular parameterized surface $\mathbf{x} : U \to \mathbb{R}^3$ such that $p_0 \in S \cap V$, $\mathbf{x}(U) = S \cap V$, and the map $\mathbf{x}$ is a homeomorphism onto its trace $S \cap V$. The last condition means that the inverse map $\mathbf{x}^{-1} : S \cap V \to U$ is continuous. The map $\mathbf{x} : U \to S \cap V \subseteq \mathbb{R}^3$ is called a **local parameterization** of S and the functions $u, v : S \cap V \to \mathbb{R}$ defined by

$$\mathbf{x}^{-1}(p) = (u(p), v(p)), \qquad p \in S \cap V$$

are called **local coordinates** on S.

5. Change of Parameters Theorem. *Let $\mathbf{x} : U_1 \to S \cap V_1 \subseteq \mathbb{R}^3$ and $\mathbf{y} : U_2 \to S \cap V_2 \subseteq \mathbb{R}^3$ be two local parameterizations and define open subsets U_{12} and U_{21} of $\mathbb{R}^2$ by*

$$U_{12} := \mathbf{x}^{-1}(S \cap V_1 \cap V_2), \qquad U_{21} := \mathbf{y}^{-1}(S \cap V_1 \cap V_2).$$

Then the map $h : U_{12} \to U_{21}$ defined by

$$h(q) = \mathbf{y}^{-1}(\mathbf{x}(q))$$

is a diffeomorphism, i.e. both h and h^{-1} are smooth.

Proof: See do Carmo pages 70-71. □

6. Definition. A subset $C \subseteq \mathbb{R}^3$ of $\mathbb{R}^3$ is called a **regular curve** iff for every point $p_0 \in C$ there is an open subset $V \subseteq \mathbb{R}^3$ and a regular parameterized curve $\alpha : I \to \mathbb{R}^3$ such that $p_0 \in C \cap V$, $\mathbf{x}(I) = C \cap V$, and the map α is a homeomorphism onto its trace $C \cap V$. The map $\alpha \to C \cap V \subseteq \mathbb{R}^3$ is called a **local parameterization** of C. (Recall from Chapter 1 that the condition that α be a *regular parameterized curve* is that $\alpha'(t) \neq 0$ for $t \in I$.)

7. Change of Parameters Theorem for Curves. *Let $\alpha : I_1 \to S \cap V_1 \subseteq \mathbb{R}^3$ and $\beta : I_2 \to S \cap V_2 \subseteq \mathbb{R}^3$ be two local parameterizations of a regular curve C and define open intervals I_{12} and I_{21} of $\mathbb{R}^3$ by*

$$I_{12} := \alpha^{-1}(C \cap V_1 \cap V_2), \qquad I_{21} := \mathbf{y}^{-1}(C \cap V_1 \cap V_2).$$

Then the map $h : I_{12} \to I_{21}$ defined by

$$h(t) = \beta^{-1}(\alpha(t)))$$

is a diffeomorphism, i.e. both h and h^{-1} are smooth.

Proof: This is Exercise 2.3-15 on page 82 of Do Carmo. $\square$

8. Example. Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = (\cos t, \sin 2t).$$

The derivative γ' never vanishes and the trace $C = \gamma(\mathbb{R})$ is a figure eight crossing itself at the origin. Let $I_1 = (\pi/2, 5\pi/2)$, $I_2 = (-\pi/2, 3\pi/2)$ and let $\alpha : I_1 \to \mathbb{R}^2$ and $\beta : I_2 \to \mathbb{R}^2$ be the restrictions of γ to the indicated intervals. Then $C = \alpha(I_1) = \beta(I_2)$ and the maps α and β are one-to-one. However there do not exist open intervals I_{12} about about $3\pi/2$ and I_{21} about $\pi/2$ such that $\alpha^{-1} \circ \beta$ is a diffeomorphism. The hypothesis of the previous theorem fails. The inverse map $\alpha^{-1} : C \cap V \to I_1$ is not a homeomorphism onto its image no matter small is the neighborhood V of the origin in $\mathbb{R}^2$

9. Theorem. Let $S \subset \mathbb{R}^3$ be a regular surface and $f : S \to \mathbb{R}$. Then the following are equivalent.

(i) *For every* local parameterization $\mathbf{x} : U \to S \cap V$ the composition $f \circ \mathbf{x} : U \to \mathbb{R}$ is a smooth function.

(ii) For every $p_0 \in S$ *there is* a local parameterization $\mathbf{x} : U \to S \cap V$ with $p_0 \in S \cap V$ such that the composition $f \circ \mathbf{x}$ smooth.

(iii) For every $p \in S$ there is an open set $V \subseteq \mathbb{R}^3$ containing p_0 and a smooth function $F : V \to \mathbb{R}$ such that $F(p) = f(p)$ for $p \in S \cap V$.

(See do Carmo page 72.) A function satisfying these equivalent properties is called **smooth**. A map $f : S \to \mathbb{R}^n$ is called **smooth** iff each of its n components is a smooth function.

10. Regular Values. Let $V \subseteq \mathbb{R}^3$ be open subset and $F : V \to \mathbb{R}$ be a smooth function. A point $p \in V$ is called a **regular point** of F iff the differential

$$dF_p := \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)_p$$

is non zero. (Here the subscript p on the right indicates that the partial derivatives are to be evaluated at p.) A real number $a \in \mathbb{R}$ is called a **regular value** of F iff every point $p \in F^{-1}(a)$ is a regular point of F.

11. Regular Value Theorem. *A subset $S \subseteq \mathbb{R}^3$ is a smooth surface if and only if for every point $p \in S$ there is an open set $V \subseteq \mathbb{R}^3$ and a smooth function $F : V \to \mathbb{R}$ such that $p \in V$, 0 is a regular value of F, and $S \cap V = F^{-1}(0)$.*

Proof. (See do Carmo page 59.) If p is a regular point of F then at least one of the three partial derivatives is non zero at p. The Implicit Function Theorem[2]

[2] Click if reading online.

states that the corresponding variable is a function of the other two in a neighborhood of p. This means that there is a regular parameterization of one of the three forms

$$\mathbf{x}(u,v) = (x(u,v), u, v), \quad \mathbf{y}(u,v) = (u, y(u,v), v), \quad \mathbf{z}(u,v) = (u, v, z(u,v)).$$

Coordinates formed this way are called <u>Monge coordinates</u>. $\square$

12. Remark. It is a theorem (page 114 of do Carmo) that a surface $S \subseteq \mathbb{R}^3$ is of form $S = F^{-1}(0)$ for some smooth $F : V \to \mathbb{R}$ having 0 is a regular value if and only if S is orientable. (See Definition 26 below for the definition of *orientable*.) This theorem requires that $S \subseteq V$ whereas Theorem 11 above is local; it only requires $S \cap V = F^{-1}(0)$. The point is that every surface is "locally orientable", but orientability is a "global condition".

13. Example. The ellipsoid is the set $F^{-1}(0)$ where

$$F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

The only point of $\mathbb{R}^3$ which is not a regular point of F is the origin and F does not vanish at the origin. The ellipsoid can be be covered by six graphs, namely

$$\begin{array}{ll} \mathbf{x}_\pm(u,v) = (\pm x(u,v), u, v), & x(u,v) := |a|\sqrt{1 - b^{-2}u^2 - c^{-2}v^2}, \\ \mathbf{y}_\pm(u,v) = (u, \pm y(u,v), v), & y(u,v) := |b|\sqrt{1 - a^{-2}u^2 - c^{-2}v^2}, \\ \mathbf{z}_\pm(u,v) = (u, v, \pm z(u,v)), & z(u,v) := |c|\sqrt{1 - a^{-2}u^2 - b^{-2}v^2}. \end{array}$$

In each case the open set $U \subseteq \mathbb{R}^2$ is defined by the condition that the quantity under the square root sign is positive (this the interior of an ellipse) and the open set $V \subseteq \mathbb{R}^3$ is the half space where the corresponding coordinate is either positive or negative as appropriate.

14. Definition. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. The **tangent vector** to S at p is a vector $\alpha'(0)$ where $\alpha : I \to \mathbb{R}^3$ is a smooth curve such that $\alpha(I) \subseteq S$, $0 \in I$, and $\alpha(0) = p$. The space of all tangent vectors to S at p is denoted by T_pS and called the **tangent space** to S at p. (See do Carmo page 83.)

15. Theorem. *Let $\mathbf{x} : U \to S \cap V \subseteq \mathbb{R}^3$ be a local parameterization of a smooth surface S, $q \in U$, and $p = \mathbf{x}(q) \in S$. Then*

$$T_pS = d\mathbf{x}_q(\mathbb{R}^2),$$

i.e. the tangent space is the image of the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$.

16. Remark. I prefer to call T_pS the **tangent space** and the translate $p + T_pS$ the **tangent plane**. The tangent space is a vector space; the tangent plane is not. On page 83 do Carmo writes "the plane $d\mathbf{x}_q(\mathbb{R}^2)$ which passes through $p = \mathbf{x}(q) \ldots$". This is incorrect as usually $p \notin d\mathbf{x}_q(\mathbb{R}^2)$. Of course, the point $p = p + 0$ lies in the tangent plane $p + T_pS$.

17. Maps between surfaces. Let $S_1, S_2 \subseteq \mathbb{R}^3$ be regular surfaces, and

$$\varphi : S_1 \to S_2$$

be a smooth map, i.e. each of its three components is a smooth function as in Theorem 9 above. An equivalent condition is that the map φ is smooth in local coordinates, i.e. for every point $p \in S$ and every local parameterization $\mathbf{y} : U_2 \to S_2 \cap V_2$ with $\varphi(p) \in S_2 \cap V_2$ there is a local parameterization $\mathbf{x} : U_1 \to S_1 \cap V_1$ such that $\varphi(U_1) \subseteq U_2$ and the map $\mathbf{y}^{-1} \circ \varphi \circ \mathbf{x} : U_1 \to U_2$ is a smooth map from the open set $U_1 \subseteq \mathbb{R}^2$ to the open set $U_2 \subseteq \mathbb{R}^2$. When $\varphi : S_1 \to S_2$ is smooth and $\alpha : I \to S_1$ is a curve in S_1 with $\alpha(0) = p$, then $\varphi \circ \alpha : I \to S_2$ is a curve with $(\varphi \circ \alpha)(0) = \varphi(p)$ so the differential

$$d\varphi_p : T_p S_1 \to T_{\varphi(p)} S_2$$

is a linear map from the tangent space to S_1 at p to the tangent space to S_2 at $\varphi(p)$. A map $\varphi : S_1 \to S_2$ is called a **diffeomorphism** iff φ is one-to-one and onto and both maps φ and φ^{-1} are smooth.

18. Inverse Function Theorem. *The differential $d\varphi_p : T_p S_1 \to T_{\varphi(p)} S_2$ is an invertible linear map if and only if f is a local diffeomorphism at p, i.e. if and only if there are open sets $S_1 \cap V_1$ and $S_2 \cap V_2$ such that $p \in S_1 \cap V_1$, $\varphi(p) \in S_2 \cap V_2$, $\varphi(S_1 \cap V_1) = S_2 \cap V_2$, and the map $\varphi : S_1 \cap V_1 \to S_2 \cap V_2$ is a diffeomorphism.*

Proof: In other words, for all $w_2 \in T_p S_2$ the equation $d\varphi_p(w_1) = w_2$ has a unique solution $w_1 \in T_p S_1$ if and only if for all $p_2 \in S_2$ near $\varphi(p)$ the equation $p_2 = \varphi(p_1)$ has a unique solution $p_1 \in S_1$ near p. A special case is where $S_1 = U_1$ and $S_2 = U_2$ are open subsets in $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$. The general case follows easily from the special case. For careful proofs of this and the other theorems (such as the Implicit Function Theorem and the Existence and Uniqueness Theorem for ODE) which Do Carmo leaves unproved see the little book *Calculus On Manifolds* by Michael Spivak. □

19. Definition. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $p \in S$. The function $I_p : T_p S \to \mathbb{R}$ defined by

$$I_p(w) := \langle w, w \rangle = |w|^2, \qquad w \in T_p S \subseteq \mathbb{R}^3$$

is called the **first fundamental form** of S at p. (See do Carmo page 92.)

20. Remark. Here do Carmo uses the notation $\langle w_1, w_2 \rangle$ for what was denoted by $w_1 \cdot w_2$ in Chapter I and calls $\langle w_1, w_2 \rangle$ the **inner product** (rather than the *dot product*) of the vectors $w_1, w_2 \in \mathbb{R}^3$. When $w_1, w_2 \in T_p S$ he sometimes writes $\langle w_1, w_2 \rangle_p$ for $\langle w_1, w_2 \rangle$. Following do Carmo I will no longer write vectors in boldface. Note that do Carmo denotes local parameterizations in bold face, but $\mathbf{x}(u, v)$ should be viewed as a *point* of $\mathbb{R}^3$ *not* a vector.

21. The First Fundamental Form in Local Coordinates. Let $S \subseteq \mathbb{R}^3$ be a regular surface and $\mathbf{x} : U \to S \cap W$ be a local parameterization. Define functions $F, E, G : U \to \mathbb{R}$ by

$$E(q) = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p, \qquad F(q) = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_p, \qquad G(q) = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p$$

for $q \in U$ and $p = \mathbf{x}(q) \in S$. Then

$$\langle \hat{p}_1, \hat{p}_2 \rangle_p = E(q)\hat{u}_1\hat{u}_2 + F(q)(\hat{u}_1\hat{v}_2 + \hat{v}_1\hat{u}_2) + G(q)\hat{v}_1\hat{v}_2$$

for $\hat{p}_i = (\hat{u}_i.\hat{v}_i) \in \mathbb{R}^2$. In particular, the first fundamental form is given by

$$I_p(\hat{p}) = E(q)\hat{u}^2 + 2F(q)\hat{u}\hat{v} + G(q)\hat{v}^2.$$

In matrix notation this formula is

$$I_p(\hat{p}) = \begin{pmatrix} \hat{u} & \hat{v} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}_q \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix}.$$

22. Example (Stereographic Projection). (See do Carmo Exercise 16 page 67.) Let $S^2 \subseteq \mathbb{R}^3$ denote the unit sphere, i.e.

$$S^2 = \{(x, y, z) \in \mathbb{R}^3,\ x^2 + y^2 + z^2 = 1\}.$$

The point $n = (0, 0, 1) \in S^2$ is called the *north pole*. The map $\pi : S^2 \setminus \{n\} \to \mathbb{R}^2$ defined by the condition

$$\pi(p) = q \iff \text{the three points } n,\ p,\ (q, 0) \text{ are collinear}$$

is called **stereographic projection**. By similar triangles (see Figure 1) we see that

$$\pi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

and the inverse map is given by $\mathbf{x}(u, v) := \pi^{-1}(u, v) = (x, y, z)$ where

$$x = \frac{2u}{u^2 + v^2 + 1}, \qquad y = \frac{2v}{u^2 + v^2 + 1}, \qquad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}.$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{-2u^2 + 2v^2 + 2}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial y}{\partial u} = \frac{-4uv}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial z}{\partial u} = \frac{-4u^2}{(u^2 + v^2 + 1)^2},$$

$$\frac{\partial x}{\partial v} = \frac{-4uv}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial y}{\partial v} = \frac{2u^2 - 2v^2 + 2}{(u^2 + v^2 + 1)^2}, \quad \frac{\partial z}{\partial v} = \frac{-4v^2}{(u^2 + v^2 + 1)^2}.$$

Hence

$$\langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle = \frac{4}{u^2 + v^2 + 1}, \qquad \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0.$$

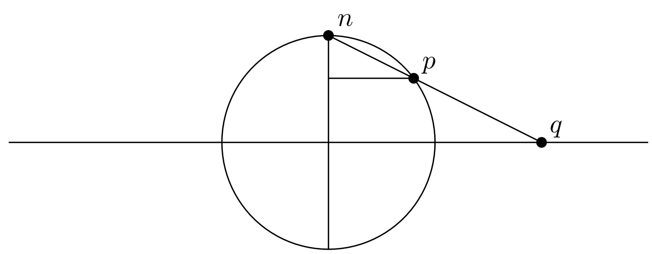

Figure 1: Stereographic Projection

Therefore for $p = (x, y, z) \in S^2 \setminus \{n\}$ and $q = (u, v) = \pi(p) \in \mathbb{R}^2$ we have

$$\langle \hat{p}_1, \hat{p}_2 \rangle = \mu(q) \langle \hat{q}_1, \hat{q}_2 \rangle$$

where

$$\hat{q}_i \in \mathbb{R}^2, \qquad \hat{p}_i = d\mathbf{x}_q(\hat{q}_i) \in T_pS^2, \qquad \mu(q) := \frac{4}{u^2 + v^2 + 1}.$$

In other words the first fundamental form satisfies $E = G$ and $F = 0$. This implies that the linear map $d\mathbf{x}_q : \mathbb{R}^2 \to T_pS^2$ preserves (cosines of) angles. A linear map which preserves angles is called **conformal**.

23. Remark. The book *Geometry and the Imagination* by David Hilbert and Stephan Cohn-Vossen (Chelsea Publishing Company, 1952) contains an elementary proof that stereographic projection is conformal on page 248. (The proof is elementary in that it doesn't use calculus.) An elementary proof can also be found online at `http://people.reed.edu/~jerry/311/stereo.pdf`. (I put a copy at `http://www.math.wisc.edu/~robbin/Do_Carmo/stereo.pdf`.)

24. Area Theorem. *Let $S \subseteq \mathbb{R}^3$ be a compact*[3] *regular surface. Then there is a unique function A which assigns a real number $A(S \cap V)$ to every open subset $S \cap V$ of S and satisfies the following two properties.*

(i) *For every local parameterization $\mathbf{x} : U \to S \cap V$ we have*

$$A(S \cap V) := \iint_U |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv$$

(ii) *If $V = V_1 \cup V_2$ and the sets $S \cap V_1$ and $S \cap V_2$ intersect only in their boundaries, then*

$$A(S \cap V) = A(S \cap V_1) + A(S \cap V_2).$$

[3] The term **compact** means *closed and bounded.*

The number $A(S)$ is called the **area** *of S.*

Proof: A careful proof of this theorem is best left for another course, but the geometric idea isn't so difficult. The key point is the change of variables formula for a double integral. (See do Carmo at the bottom of page 97.) This formula says that

$$\iint_{U_1} |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv = \iint_{U_2} |\mathbf{y}_u \wedge \mathbf{y}_v| \, du \, dv$$

if $\mathbf{x} : U_1 \to S \cap V$ and $\mathbf{y} : U_2 \to S \cap V$ are two local parameterizations with the same trace. i.e. $\mathbf{x}(U_1) = \mathbf{y}(U_2)$. Then we must show that S can be covered by open sets which overlap only in their boundaries. (A precise definition of *boundary* must be given.) Finally we must prove the addition formula in part (ii).

The formula in part (i) is plausible. Imagine that the set U is broken up into a large number of very small rectangles. Each rectangle has area $du\,dv$. The image of this rectangle under the map $\mathbf{x}$ will be approximately a parallelogram with edge vectors $\mathbf{x}_u \, du$ and, $\mathbf{x}_v \, dv$ and the area of this parallelogram is roughly

$$dA = |\mathbf{x}_u \wedge \mathbf{x}_v| \, du \, dv.$$

Now $|\mathbf{x}_u \wedge \mathbf{x}_v| = |\sin\theta| \, |\mathbf{x}_u| \, |\mathbf{x}_v|$ where θ is the angle from $\mathbf{x}_u$ to $\mathbf{x}_v$. But this is the area of the tiny parallelogram. Adding up all these tiny areas gives the total area as an integral. In terms of the first fundamental form the area element in local coordinates is

$$dA = \sqrt{EG - F^2} \, du \, dv.$$

This is a consequence of the formulas

$$\langle w_1, w_2 \rangle = |w_1| \, |w_2| \cos\theta, \qquad |w_1 \wedge w_2| = |w_1| \, |w_2| \, |\sin\theta|$$

for the inner product and wedge product of two vectors $w_1, w_2 \in \mathbb{R}^3$. $\square$

25. Example. As an example we will prove the formula

$$A(S^2) = 4\pi$$

in two different ways. A parameterization of the upper hemisphere is

$$\mathbf{x}(u, v) = (u, v, z(u, v)), \qquad z(u, v) := \sqrt{1 - u^2 - v^2}.$$

The coordinate vectors are

$$\mathbf{x}_u = \begin{pmatrix} 1 \\ 0 \\ \dfrac{-u}{\sqrt{1 - u^2 - v^2}} \end{pmatrix}, \qquad \mathbf{x}_v = \begin{pmatrix} 0 \\ 1 \\ \dfrac{-v}{\sqrt{1 - u^2 - v^2}} \end{pmatrix},$$

so

$$\mathbf{x}_u \wedge \mathbf{x}_v = \begin{pmatrix} \dfrac{v}{\sqrt{1 - u^2 - v^2}} \\ \dfrac{-u}{\sqrt{1 - u^2 - v^2}} \\ 1 \end{pmatrix}, \qquad |\mathbf{x}_u \wedge \mathbf{x}_v| = \frac{1}{\sqrt{1 - u^2 - v^2}}.$$

To evaluate the integral we use the change of variables

$$(0,1) \times (0,2\pi) \to \{(u,v),\ u^2+v^2<1\} : (r,\theta) \mapsto (u,v) = (r\cos\theta, r\sin\theta)$$

so $du\,dv = \dfrac{\partial(u,v)}{\partial(r,\theta)}\,dr\,d\theta$ where

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \det \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{\partial u}{\partial \theta} \\ \dfrac{\partial v}{\partial r} & \dfrac{\partial v}{\partial \theta} \end{pmatrix} = r$$

so

$$\int_{u^2+v^2<1} |\mathbf{x}_u \wedge \mathbf{x}_v|\,du\,dv = \int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{1-r^2}} = 2\pi \int_0^1 \frac{ds}{2\sqrt{s}} = 2\pi.$$

The parameterization $(u,v) \mapsto (u,v,-z(u,v))$ of the lower hemisphere gives the same answer and the two hemispheres intersect only in their common boundary (the unit circle in the (x,y)-plane) so the area of S^2 is 4π.

A second way to prove $A(S^2) = 4\pi$ is to use spherical coordinates

$$\mathbf{x}(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi).$$

Here $\mathbf{x} : (0,2\pi)\times(0,\pi) \to S^2 \cap V$ where $V = \{(x,y,z) \in \mathbb{R}^3,\ x \neq 1,\ z \neq \pm 1\}$. Then

$$\mathbf{x}_\theta = \begin{pmatrix} -\sin\theta\cos\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix}, \qquad \mathbf{x}_\varphi = \begin{pmatrix} -\cos\theta\sin\varphi \\ -\sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix},$$

so

$$\mathbf{x}_\theta \wedge \mathbf{x}_\varphi = \begin{pmatrix} \cos\theta\cos^2\varphi \\ \sin\theta\cos^2\varphi \\ \cos\varphi\sin\varphi \end{pmatrix}, \qquad |\mathbf{x}_\theta \wedge \mathbf{x}_\varphi| = |\cos\varphi|.$$

Now $S^2 \cap V$ intersects itself only in its boundary (which is a semicircle) so

$$A(S^2) = \int_0^\pi \int_0^{2\pi} |\cos\varphi|\,d\theta\,d\varphi = 4\pi.$$

26. The Unit Normals. For a two dimension vector subspace $W \subseteq \mathbb{R}^3$ there are exactly two unit vectors $n \in \mathbb{R}^3$ which are perpendicular to every vector in W, i.e. such that $|n| = 1$ and $\langle n, w\rangle = 0$ for $w \in W$. If n is one of these two vectors then $-n$ is the other one. In particular, when $W = T_pS$ is the tangent space at a point p to a regular surface $S \subseteq \mathbb{R}^3$ there are exactly two vectors $N(p)$ such that $|N(p)| = 1$ and $\langle N(p), w\rangle = 0$ for all $w \in T_pS$. If $\mathbf{x} : U \to S \cap V$ is a local parameterization of S and $p = \mathbf{x}(q) \in S$ where $q \in U$, then these two unit normal vectors are

$$N(p) = \pm\left(\frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}\right)_q.$$

27. Definition. A regular surface is said to be **orientable** iff there is a smooth map $N : S \to S^2$ such that

$$\langle N(p), w\rangle = 0, \qquad \forall w \in T_pS.$$

Such a map determines an **orientation** on each tangent space: an ordered basis $w_1, w_2 \in T_pS$ is positively oriented iff $\langle N(p), w_1 \wedge w_2\rangle > 0$. The vector field N is called the **unit normal** to the oriented surface S and the map $N : S \to S^2$ is called the **Gauss map**.

28. Remark. It is a difficult theorem that a compact regular surface S is orientable and the open set $\mathbb{R}^3 \setminus S$ has two connected components, one bounded and the other unbounded. In this case one chooses the orientation so that the normal vector N points into the unbounded component. This N is called the **outward unit normal** vector. For example, when $S = S^2$ the bounded component is the open ball $\{(x, y, z) \in R^3, x^2 + y^2 + z^2 < 1\}$ and the unbounded component is the open set $\{(x, y, z) \in R^3, x^2 + y^2 + z^2 > 1\}$. The outward unit normal for S^2 is $N(p) = p$ so the Gauss map is the identity map.

29. The Möbius Strip. (See do Carmo page 106.) This is the image S of the map $\mathbf{x} : \mathbb{R} \times (-1, 1) \to \mathbb{R}^3$ defined by

$$\mathbf{x}(\theta, r) = \mathbf{z}(\theta)+r\mathbf{n}(\theta), \quad \mathbf{z}(\theta) = (2\cos 2\theta, 2\sin 2\theta, 0), \quad \mathbf{n}(\theta) = (\sin\theta, \sin\theta, \cos\theta).$$

The curve $\mathbf{z}$ has period π and the curve $\mathbf{n}$ has period 2π. Note that the line segment $\ell(\theta)$ connecting the two points $\mathbf{x}(\theta, \pm 1)$ lies in S and if $0 < |\theta_1 - \theta_2| < \pi$ the two line segments $\ell(\theta_1)$ and $\ell(\theta_2)$ do not intersect. The two line segments $\ell(\theta)$ and $\ell(\theta + \pi)$ are equal as sets but they have opposite orientations. Since $\mathbf{x}_\theta = \mathbf{z}'(\theta) + r\mathbf{n}'(\theta)$ and $\mathbf{x}_r = \mathbf{n}(\theta)$ we get

$$(\mathbf{x}_\theta \wedge \mathbf{x}_r)(\theta, 0) = \mathbf{z}'(\theta) \wedge \mathbf{n}(\theta)$$

so $\mathbf{x}(\theta + \pi, 0) = \mathbf{x}(\theta, 0)$ but $\mathbf{x}_\theta \wedge \mathbf{x}_r(\theta + \pi, 0) = -\mathbf{x}_\theta \wedge \mathbf{x}_r(\theta, 0)$. Hence there is no continuous unit normal so the Möbius strip is not orientable.

The Gauss map

JWR

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1. Let $C \subset \mathbb{R}^3$ be a curve and $p \in C$. Let $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a parameterization of C by arc length centered at p, i.e.

$$\|\alpha'(s)\|^2 = 1, \qquad \alpha(0) = p.$$

The vector $\alpha''(0)$ is called the **curvature vector** at p. Differentiating shows that $\langle \alpha'', \alpha' \rangle = 0$ so the curvature vector is orthogonal to the tangent vector $\alpha'(0)$ to the curve at p. Reversing the orientation of the curve (i.e. replacing s by $-s$) reverses the direction of the tangent vector but leaves the curvature vector unchanged.

2. Let $S \subset \mathbb{R}^3$ be an oriented surface. The **Gauss map** is the map $N : S \to S^2$ which assigns to $p \in S$ the unit normal. There are two unit normals ($-N$ is the other one); the meaning of the word *oriented* is that we have chosen one. Thus[1]

$$\|N(p)\| = 1, \qquad \langle N(p), \mathbf{v} \rangle = 0 \text{ for } \mathbf{v} \in T_pS. \qquad \text{page 136}$$

The **first fundamental form** assigns to each $p \in S$ the quadratic form $I_p : T_pS \to \mathbb{R}$ defined by

$$I_p(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 \qquad \text{page 92}$$

It assigns to each tangent vector $\mathbf{v} \in T_pS \subset \mathbb{R}^3$ the square of its length. The **second fundamental form** is defined by

$$II_p(\mathbf{v}) = \langle N(p), \alpha''(0) \rangle, \qquad \mathbf{v} = \alpha'(0)$$

where $\alpha : (\varepsilon, \varepsilon) \to S$ is a curve whose tangent vector at p is $\mathbf{v}$. Equation (†) below says that $II_\alpha(\alpha')$ is the normal component of the curvature vector α''.

3. Lemma. *The second fundamental form is independent of the choice of curve α used to define it.*

Proof. Since $\alpha(s) \in S$ we have $\alpha'(s) \in T_{\alpha(s)}S$ and hence $\langle N(\alpha(s), \alpha'(s) \rangle = 0$. Differentiating gives

$$\langle dN_p(\alpha'(0), \alpha'(0) \rangle + \langle N(p), \alpha''(0) \rangle$$

This shows that

$$II_p(\mathbf{v}) = - \langle dN_p(\mathbf{v}), \mathbf{v} \rangle \text{ for } \mathbf{v} \in T_pS \qquad \text{page 141}$$

is independent of the second derivative. □

[1] All page references are to the Do Carmo text.

4. Lemma. *The derivative* $dN_p : T_pS \to T_{N(p)}S^2$ *of the Gauss map is a map from a vector space to itself, i.e.*

$$T_pS = T_{N(p)}S^2$$

for $p \in S$.

Proof. $T_pS = N(p)^\perp$ and $T_wS^2 = w^\perp$ for $w \in S^2$. $\square$

5. Lemma. *The derivative* $dN_p : T_pS \to T_pS$ *is self adjoint, i.e.*

$$\langle dN_p(\mathbf{u}), \mathbf{v}\rangle = \langle \mathbf{u}, dN_p(\mathbf{v})\rangle$$

for $\mathbf{u}, \mathbf{v} \in T_pS$.

Proof. See Proposition 1 page 140. Choose a parameterization $\mathbf{x} : U \to S$ with

$$\mathbf{x}(0,0) = p, \qquad \mathbf{x}_u(0,0) = \mathbf{u}, \qquad \mathbf{x}_v(0,0) = \mathbf{v}.$$

Here (u, v) are the standard coordinates on the open set $U \subset \mathbb{R}^2$ and the subscripts u and v indicate partial differentiation.[2] Since $N(\mathbf{x}) \perp T_{\mathbf{x}}S$ and $\mathbf{x}_u, \mathbf{x}_v \in T_{\mathbf{x}}S$ we have

$$\langle N, \mathbf{x}_u\rangle = \langle N, \mathbf{x}_v\rangle = 0$$

so

$$\langle N_v, \mathbf{x}_u\rangle + \langle N, \mathbf{x}_{uv}\rangle = \langle N_u, \mathbf{x}_v\rangle + \langle N, \mathbf{x}_{vu}\rangle = 0. \qquad (*)$$

The lemma follows from $\mathbf{x}_{uv} = \mathbf{x}_{vu}$. $\square$

6. Remark. Let $\alpha : (-\varepsilon, \varepsilon) \to S$ be a curve in S parameterized by arclength. By the geometric definition of the cross product, the vectors $N, \alpha', N \wedge \alpha'$ are orthonormal at each point $\alpha(s)$. The vector α' is a unit vector tangent to S (at α) and $N(\alpha)$ is a unit vector normal to S so $N \wedge \alpha'$ is a unit vector tangent to S and is orthogonal to both N and α'. Since $\|\alpha'\| = 1$ we also have $\langle \alpha', \alpha''\rangle = 0$. Hence the curvature vector can be written as

$$\alpha'' = k_n N + k_g(N \wedge \alpha'), \qquad k_n := \langle \alpha'', N\rangle, \qquad k_g := \langle \alpha'', N \wedge \alpha'\rangle \qquad (\dagger)$$

The coefficient k_n is called the **normal curvature** and coefficient k_g is called the **geodesic curvature**. By definition

$$II_\alpha(\alpha') = -\langle \alpha'', N(\alpha)\rangle = -k_n.$$

By the Pythagorean theorem

$$k^2 = k_n^2 + k_g^2.$$

See page 249. A **geodesic** in S is a curve whose geodesic curvature is zero, i.e. whose curvature vector is normal to S.

[2]However the subscript p in the expression $dN_p(\mathbf{u})$ indicates that that the derivative is to be evaluated at p.

7. Remark. The curvature vector is the acceleration from classical mechanics so a particle moving in S and acted on by a force which is perpendicular to to S (and no other forces) moves along a geodesic.

8. Definition. The eigenvalues k_1, k_2 of dN_p are called the **principal curvatures** and the determinant

$$K := \det(dK_p) = k_1 k_2$$

is called the **Gauss curvature**. The average value

$$H := \frac{k_1 + k_2}{2}$$

of the principal curvatures is the called the **mean curvature**. Thus $\lambda = k_1$ and $\lambda = k_2$ are the two solutions of the characteristic equation

$$\lambda^2 + 2H\lambda + K = 0.$$

9. Remark. If dA denotes the area of an infinitesimal region on S containing the point p, then $K(p)\,dA$ is the area of the image of that infinitesimal region under the Gauss map. Thus $K(p)$ is the analog for surfaces of the curvature $k = d\theta/ds$ of a plane curve.

10. Let $U \subset \mathbb{R}^2$ be open and $\mathbf{x} : U \to S$ be a parameterization. The unit normal is

$$N = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{\|\mathbf{x}_u \wedge \mathbf{x}_v\|}. \qquad \text{page 135}$$

A curve $\alpha : (-\varepsilon, \varepsilon) \to S$ can be written

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$

where $(u(t), v(t)) \in U$. In these coordinates the fundamental forms are given by

$$I_\alpha(\alpha') = E(u')^2 + 2Fu'v' + G(v')^2, \qquad \text{page 92}$$

$$II_\alpha(\alpha') = e(u')^2 + 2fu'v' + g(v')^2 \qquad \text{page 154}$$

where

$$\begin{array}{lll} E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle, & F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle, & G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\ e = -\langle N_u, \mathbf{x}_u \rangle, & f = -\langle N_u, \mathbf{x}_v \rangle, & g = -\langle N_v, \mathbf{x}_v \rangle. \end{array}$$

are functions on U. The subscript on N means partial differentiation so

$$N_u = dN_{\mathbf{x}}(\mathbf{x}_u), \qquad N_v = dN_{\mathbf{x}}(\mathbf{x}_v).$$

By $(*)$ f can be written four ways.

11. Weingarten Equations.

$$N_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v. \qquad N_v = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v \qquad \text{page 154}$$

where

$$a_{11} = \frac{fF - eG}{EG - F^2}, \qquad a_{12} = \frac{gF - fG}{EG - F^2},$$
$$a_{21} = \frac{eF - fE}{EG - F^2}, \qquad a_{22} = \frac{fF - gE}{EG - F^2}. \qquad \text{page 155}$$

12. Corollary. *The Gauss curvature is given by*

$$K = \frac{eg - f^2}{EG - F^2}$$

and the mean curvature is given by

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}.$$

13. Suppose that the surface S is a graph, i.e. it is defined by an equation

$$z = h(x, y).$$

The tangent space at $p = (x, y, z) \in S$ is the graph of dh i.e. the set of all vectors (x', y', z') such that

$$z' = h_x(x, y)x' + h_y(x, y)y'.$$

The vector

$$N = \frac{(-h_x, -h_y, 1)}{\|N\|}, \qquad \|N\| = \sqrt{h_x^2 + h_y^2 + 1}$$

is one of the two unit normal vectors to S. There is an obvious parameterization $\mathbf{x}(u, v) = (x, y, z)$ where

$$x = u, \qquad y = v, \qquad z = h(u, v). \qquad (\#)$$

For this parameterization

$$\mathbf{x}_u = (1, 0, h_x), \qquad \mathbf{x}_v = (0, 1, h_y)$$

so

$$E = 1 + h_x^2, \qquad F = h_x h_y, \qquad G = 1 + h_y^2,$$
$$e = \frac{h_{xx}}{\|N\|}, \qquad f = \frac{h_{xy}}{\|N\|}, \qquad g = \frac{h_{yy}}{\|N\|},$$
$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{\|N\|^2},$$
$$2H = \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{\|N\|^{3/2}}.$$

14. Theorem. *If $K(p) > 0$, then S lies to one side of $p + T_pS$ near p. If $K(p) < 0$, then S intersects $p + T_pS$.*

Proof. Choose coordinates on $\mathbb{R}^3$ so that $p = (0,0,0)$, $T_pS =$ the xy-plane. Then S is a graph near p with equation $z = h(x,y)$ and $h(0,0) = h_x(0,0) = h_y(0,0) = 0$ and $d^2h(0,0)$ is the second fundamental form. Rotate the (x,y) plane so that $(1,0)$ and $(0,1)$ are eigenvectors of Hessian matrix

$$d^2h(0,0) = \begin{pmatrix} h_{xx}(0,0) & h_{xy}(0,0) \\ h_{yx}(0,0) & h_{yy}(0,0) \end{pmatrix}.$$

Then $x_{xy}(0,0) = h_{yx}(0,0) = 0$ so the principle curvatures are $k_1 = h_{xx}(0,0)$ and $k_2 = h_{yy}(0,0)$. The second fundamental form is $k_1x^2 + k_2y^2$. Then

$$h(x,y) = k_1x^2 + k_2y^2 + \text{higher order terms}.$$

See Proposition 3 in section 2-2 on page 63 and problem 26 on page 91. □

15. Remark. The *Implicit Function Theorem* says that if $N(p)$ does not lie in the xy-plane then p lies in the image of a local parameterization as in equation (#). This is Proposition 3 in section 2-2 on page 63. Since $N(p)$ cannot lie in all three coordinate planes it is always possible to choose two of the three coordinates x, y, z to parameterize the surface (near p) as a graph. For example, the unit sphere is covered by six parameterizations

$$\begin{array}{ccc} z = \sqrt{1-x^2-y^2}, & y = \sqrt{1-x^2-z^2}, & x = \sqrt{1-y^2-z^2}, \\ z = -\sqrt{1-x^2-y^2}, & y = -\sqrt{1-x^2-z^2}, & x = -\sqrt{1-y^2-z^2}. \end{array}$$

Other local parameterizations of the unit sphere are by cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = \sqrt{1-r^2}$$

(this parameterizes the northern hemisphere), by spherical coordinates

$$x = \cos\theta\sin\varphi, \qquad y = \sin\theta\sin\varphi. \qquad z = \cos\varphi$$

(this parameterizes everything but the north and south poles) and stereographic projection

$$x = \frac{2u}{1+u^2+v^2}, \qquad y = \frac{2v}{1+u^2+v^2}, \qquad z = \frac{1-u^2-v^2}{1+u^2+v^2}$$

(this parameterizes everything but the south pole $(0,0,-1)$. See exercise 16 page 67.) The Gauss curvature of the unit sphere is (obviously) identically equal to one as the Gauss map is the identity map.

16. The point $(\cos(u \pm \nu), \sin(u \pm \nu), \pm 1)$ lies in the plane $z = \pm 1$. When $\nu = 0$ these points lie on the same vertical line but for $\nu > 0$ the upper one has been rotated clockwise and the lower one has been rotate counter clockwise. The line connecting these two points has parametric equations

$$x = x_0 + v\xi, \qquad y = y_0 + v\eta, \qquad z = v$$

where $(x_0, y_0, 0)$ is the midpoint of the line segment connecting them and $(2\xi, 2\eta, 2)$ is the vector from the lower point to the upper, i.e.

$$\begin{aligned} x_0 &= \tfrac{1}{2}(\cos(u+\nu) + \cos(u-\nu)) = \cos u \cos\nu, \\ y_0 &= \tfrac{1}{2}(\sin(u+\nu) + \sin(u-\nu)) = \sin u \cos\nu \end{aligned}$$

and

$$\begin{aligned} \xi &= \tfrac{1}{2}(\cos(u+\nu) - \cos(u-\nu)) = -\sin u \sin\nu, \\ \eta &= \tfrac{1}{2}(\sin(u+\nu) - \sin(u-b)) = \cos u \sin\nu. \end{aligned}$$

Since $x_0\xi + y_0\eta = 0$ we get

$$x^2 + y^2 = \cos^2\nu + v^2\sin^2\nu = a^2 + b^2z^2$$

where $a = \cos\nu$ and $b = \sin\nu$. This is the equation of a hyperboloid of one sheet. Replacing ν by $-\nu$ gives the same equation so the hyperboloid of one sheet contains two lines though every point. The tangent plane at any point intersects the hyperboloid in these two lines so the hyperboloid has negative Gauss curvature.

17. The equation $x^2 + y^2 = z^2 + 1$ defines a hyperboloid of one sheet, and the equation $x^2 + y^2 = z^2 - 1$ defines a hyperboloid of two sheets. The latter has positive Gauss curvature and therefore contains no lines.

18. Stereographic projection $\mathbb{R}^2 \to S^2$ is defined by the condition that the three points

$$s = (0, 0, -1), \qquad p = (x, y, z), \qquad w = (u, v, 0), \qquad x^2 + y^2 + z^2 = 1$$

are collinear. It covers the entire sphere *except for the south pole* $s = (0, 0, -1)$ in a one-one way. The analogous condition that the three points

$$s = (0, 0, -1), \qquad p = (x, y, z), \qquad w = (u, v, 0), \qquad x^2 + y^2 - z^2 = -1$$

be collinear be used to parameterize the upper sheet of the hyperboloid of one sheet by the unit disk $u^2 + v^2 < 1$. In this example the parameterization covers the whole upper sheet in a one-one way. (The south pole $s = (0, 0, -1)$ is on the lower sheet.)