We will see a proof of Laver's theorem proving Fraïssé's conjecture. The proof we present is due to van Engelen, Miller, and Steel.

A reminder

We use the following definition of BQO.

- Given Q a qo, say that a map

  $f : [X]^{\mathbb{N}} \longrightarrow Q$ for some $X \in [\mathbb{N}]^{\mathbb{N}}$ is a Q-multisequence.

- Say that f is weak Borel if $|\operatorname{im} f| \leq \aleph_0$ and $f^{-1}(\{q\})$ is Borel for all $q \in Q$.

- f is bad if $\forall Z \in [X]^{\mathbb{N}}$ $f(Z) \not\leq_Q f(Z_+)$ where $Z_+ = Z \setminus \min Z$.

- Say that Q is BQO if there are no bad, weak Borel, Q-multisequence.

Note: Q BQO $\Rightarrow$ Q WQO.

Indeed, by contraposition if $(q_n)_n$ is a bad Q-sequence then $f: [\mathbb{N}]^{\mathbb{N}} \to Q$, $X \mapsto q_{\min X}$ is a bad, weak Borel Q-multisequence (it is actually loc. constant).

The minimal bad multisequence lemma

- A partial ranking of $\leq_Q$ is a qo $\leq'$ on Q satisfying: $\forall p,q$ $p \leq' q \Rightarrow p \leq_Q q$ and $\leq'$ is well-founded.
- A bad Q-multisequence f is $\leq'$-minimal if for all Q-multiseq. g, if $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ and $\forall X \in \operatorname{dom} f \cap \operatorname{dom} g$ we have $g(X) <' f(X)$ then g is good.

Thm (Nash-Williams) Let Q be a qo, $f_0 : [X_0]^{\mathbb{N}} \to Q$ a bad, weak Borel Q-multisequence for some $X_0 \in [\mathbb{N}]^{\mathbb{N}}$, and $\leq'$ be a partial ranking of $\leq_Q$.
Then there is $Z \in [X]^{\mathbb{N}}$ and a $\leq'$-minimal bad weak Borel $g : [Z]^{\mathbb{N}} \to Q$ st $g(A) \leq' f(A)$ for all $A \in [Z]^{\mathbb{N}}$.

First a combinatorial fact, left mostly as an exercise.
For X, Y in $[\mathbb{N}]^{\mathbb{N}}$ note $X \subseteq^* Y$ if $X \setminus Y$ is finite.

Fact: If $(X_\alpha)_{\alpha<\omega_1}$ is an $\subseteq^*$-decreasing chain in $[\mathbb{N}]^{\mathbb{N}}$ then there is $\Lambda \in [\omega_1]^{\omega}$ and $Z \in [\mathbb{N}]^{\mathbb{N}}$ ST
$$Z \subseteq \bigcap_{\alpha\in\Lambda} X_\alpha.$$

Sketch. Use the pigeon-hole principle on an ω-partition of ω_1 to inductively build $F_n \in [\omega_1]^n$, $S_n \in [\omega]^n$ and $A_n \in [\omega_1]^{\omega_1}$ ST $F_n \subseteq F_{n+1}$, $S_n \subseteq S_{n+1}$, $A_n \supseteq A_{n+1}$ and $\forall \alpha \in F_n \cup A_n$ we have $S_n \subseteq X_\alpha$. □.

Suppose that there is no g as in the statement, we build for all $\alpha < \omega_1$ a bad weak Borel $f_\alpha : [X_\alpha]^{\mathbb{N}} \to Q'$ ST $\forall \alpha < \beta$ $X_\beta \subseteq^* X_\alpha$ and $\forall Z \in [X_\beta \cap X_\alpha]^{\mathbb{N}}$ $f_\beta(Z) <' f_\alpha(Z)$.

The successor is just the absurd hypothesis. We do the limit step for some $\lambda < \omega_1$ limit.

First take $Y \in [X_0]^{\mathbb{N}}$ ST $\forall \alpha < \lambda$ $Y \subseteq^* X_\alpha$. (exercise)
Note that by well-foundedness of $\leq'$, for all $X \in [Y]^{\mathbb{N}}$

the set $\{\alpha <' \alpha \mid X' \subseteq X_\alpha\}$ is *finite* by induction hyp.
Call α_X its maximum.

Consider now $g : [Y]^{\mathbb{N}} \longrightarrow Q$, $X \longmapsto f_{\alpha_X}(X)$.

- g is weak Borel: $\operatorname{im} g \subseteq \bigcup_{\alpha < \alpha} \operatorname{im} f_\alpha$ so it's countable and $X \in g^{-1}(\{q\})$ iff $\exists \alpha <' \alpha$ $X \in f_\alpha^{-1}(\{q\}) \cap [X_\alpha]^{\mathbb{N}}$ and $\forall \beta \geq \alpha$, $\beta < \alpha$ $X \notin [X_\beta]^{\mathbb{N}}$.
- g is bad. Note that since $X_+ \subseteq X$ we have $\alpha_{X_+} \geq \alpha_X$ so if $g(X) \leq_Q g(X_+)$ then f_{α_X} would be good:

$$f_{\alpha_X}(X) = g(X) \leq_Q g(X_+) = f_{\alpha_{X_+}}(X_+) <' f_{\alpha_X}(X_+).$$

Finally apply the absurd hyp. to g to get f_α. $\square$.

First application: Q BQO $\Rightarrow Q^{<ON}$ BQO.

Call $Q^{<ON}$ the class of ordinal Q sequences, quasi-ordered by $s \leq_{Q^{<ON}} t$ if there is a 1-1 increasing map $\sigma : \lg(s) \rightarrow \lg(t)$ st $s(\alpha) \leq_Q t(\sigma(\alpha))$ $\forall \alpha < \lg(s)$.

Fact: Q qo, s, t in $Q^{<ON}$. If $s \not\leq_{Q^{<ON}} t$ then

there exists $\Theta < \mathrm{lg}(s)$ st
$s \restriction \Theta \leq_{Q^{<\omega}} t$ but $s \restriction \Theta+1 \not\leq_{Q^{<\omega}} t$.

Proof is an exercise.

Thm (Nash-Williams) Q is a BQO $\Rightarrow Q^{<\omega}$ is a BQO.

Pf: For s, t in $Q^{<\omega}$ set $s <' t$ if $s \leq_{Q^{<\omega}} t$ and $\mathrm{lg}(s) < \mathrm{lg}(t)$. $\leq'$ is a partial ranking.

Suppose $f: [Z]^{\omega} \to Q$ is a minimal bad weak Borel Q-multiseq. Apply the fact to find Θ_X st for all $X \subseteq Z$ we have $f(X) \restriction \Theta_X \leq f(X_+)$ but
$f(X) \restriction \Theta_X + 1 \not\leq f(X_+)$

Note that $g: [Z]^{\omega} \to Q$, $X \mapsto f(X) \restriction \Theta_X$
is good by minimality of f.

By Galvin-Prikry we can assume that for all X
either $f(X)(\Theta_X) \leq_Q f(X_+)(\Theta_{X_+})$ or
$f(X)(\Theta_X) \not\leq_Q f(X_+)(\Theta_{X_+})$.

In the former case by goodness of g we would have $f(x)\restriction\Theta_x+1 \leq_{Q\text{-}ON} f(X_+)\restriction\Theta_{X_+}+1 \leq_{Q\text{-}ON} f(X_+)$ contradicting the choice of Θ_x.
So we have to be in latter case, and Q is not BQO $\square$

The Q-trees are the Q-labelled trees $\ell : T \to Q$ where T is a tree of height at most ω.

Say that $\ell \leq_{Q\text{-trees}} \ell'$ if there is a $\wedge$-embedding

$\sigma : T \to T'$ s.t. $\ell(s) \leq_Q \ell'(\sigma(s)) \ \forall s \in T$.

<u>Thm</u> (NW) $\boxed{Q \text{ BQO} \Rightarrow Q\text{-trees BQO}}$

We won't see the proof. The crucial part is getting the partial ranking (unsurprisingly).

<u>Def</u>: Given a class $\mathcal{C}$ of structures w/ morphisms, say that $\mathcal{C}$ <u>preserves BQO</u> if for all BQO Q, the Q-labelled $\mathcal{C}$-structures are still BQO, where Q-labelled $\mathcal{C}$-structures are maps

$\ell : M_\ell \to Q$ where M_ℓ is a $\mathcal{C}$-structure, and
$\ell \leq_{Q\text{-}\mathcal{C}} \ell'$ iff there is a morphism $\sigma : M_\ell \to M_{\ell'}$
st $\ell(x) \leq_Q \ell'(\sigma(x))$.

<u>Ex</u> We have just seen that $(ON, \leq_i)$ preserves BQO, and (Trees, 1-emb) as well.

<u>Note</u>: Applying the def w/ $Q = (1, =)$ we see that preserving BQO $\Rightarrow$ BQO.
The converse is false! Look at $\mathcal{C} = \{(\mathbb{R}, \leq)\}$ and as morphisms order-embeddings. Then $(2, =)$ is BQO but $2^{\mathbb{R}}$ is not... And since $\mathcal{C}$ has only one structure, $\mathcal{C}$ is BQO but does not preserve BQO.

<u>WARNING</u>: A confusion arose during the twelfth lecture. I had in mind a proof of Laver's Theorem which used that ON preserves BQO! It turns out that a stronger property is needed. The gap was filled at the end of the lecture, after the registration. Here are the notes for it.

Def : We say that a class of structures $\mathcal{C}$ reflects bad multisequences if for any Q qo and bad weak Borel Q-$\mathcal{C}$ multisequence $f : [X]^{\mathbb{N}} \longrightarrow Q$-$\mathcal{C}$ ($X \in [\mathbb{N}]^{\mathbb{N}}$), there is a $Y \in [X]^{\mathbb{N}}$ such that $\forall Z \in [Y]^{\mathbb{N}}$ there is a $h_Z \in \mathrm{dom}(f(Z))$ ST

$$\begin{aligned} g : [Y]^{\mathbb{N}} &\longrightarrow Q \\ Z &\longmapsto f(Z)(h_Z) \end{aligned}$$

is a bad weak Borel Q-multisequence.

Note that $\mathcal{C}$ reflects bad multisequence $\Rightarrow$ $\mathcal{C}$ preserves BQO. The converse is — to my knowledge — open.

Thm (NW) $\boxed{\text{ON reflects bad multisequences.}}$

Pf : Let Q be a qo, consider $\leq'$ on $Q^{<ON}$ as follows: $s \leq' t$ if there exists $\theta \leq lg(t)$ ST $s = t \restriction \theta$. Take a minimal bad weak Borel $Q^{<ON}$-multiseq. f. For all $X \in \mathrm{dom} f$ use the fact on $f(X), f(X_+)$ to get θ_X ST $f(X) \restriction \theta_X \leq_{Q^{<ON}} f(X_+)$ but $f(X) \restriction \theta_X + 1 \not\leq_{Q^{<ON}} f(X_+)$

By minimality of f, $g : X \longmapsto f(X) \restriction \theta_X$ is good, so apply GP to get $Y \in \mathrm{dom} f$ ST $g \restriction [Y]^{\mathbb{N}}$ is perfect, that is $\forall X \subseteq Y$ $g(X) \leq_{Q^{<ON}} g(X_+)$.

Now by GP again we can find $Z \in [Y]^{\mathbb{N}}$ st either $\forall X \in [Z]^{\mathbb{N}}$

[illegible]
$f(x)\restriction(\theta_x) \leq_Q f(x_+)\restriction(\theta_{x_+})$ or $\forall x \in [Z]^{\mathbb{N}}$ $f(x)\restriction(\theta_x) \not\leq_Q f(x_+)\restriction(\theta_{x_+})$

The former possibility is impossible since $f(x)\restriction\theta_x+1 \not\leq_{Q^{<ON}} f(x_+)$

The latter thus must hold, yielding a bad Q-multisequence. $\square$

Fraïssé's conjecture, aka Laver's Theorem

Def: Let SCAT be the class of scattered linear orders, that is those which do not contain a copy of $\mathbb{Q}$, so $\mathbb{Q} \not\leq_i K$.

Denote by S_0 the class of singleton orders. If S_α is defined denote $S_{\alpha+1}$ the class of ordinal sums and reverse ordinal sums of elements of S_α, and if S_α is defined for all $\alpha < \lambda$ limit denote $S_\lambda = \bigcup S_\alpha$.
Finally $S = \bigcup_{\alpha \in ON} S_\alpha$.

For $K \in S$ set $rk_H(K) = \min(\alpha \mid K \in S_\alpha)$ the Haussdorf rank of K.

Thm (Hausdorff) $\boxed{\mathcal{S} = SCAT}$

Pf sketch: $\mathcal{S} \subseteq SCAT$ by induction on rk_H.

To see that $SCAT \subseteq \mathcal{S}$, take K any linear order, and set $x \sim y$ if $[x,y]_K \in \mathcal{S}$.

There are 2 possibilities:

Case 1: $K/\sim$ is a singleton, then $K \in \mathcal{S}$

Case 2: There are x, y st $x \not\sim y$. But then there is z st $x \leq_K z \leq_z y$ and $x \not\sim z$ and $y \not\sim z$. Otherwise $[x]_\sim$ and $[y]_\sim$ being two intervals of K with no element in-between, $[x]_\sim + [y]_\sim \in \mathcal{S}$ because $[x]_\sim$ and $[y]_\sim \in \mathcal{S}$. So we would have $x \sim y$, imp.

Finally $\leq_K$ induces a dense order on $K/\sim$, so $\mathbb{Q} \leq_i K/\sim \leq_i K$ and $K \notin SCAT$. $\square$

Thm (Laver) $\boxed{\text{SCAT reflects bad multisequences}}$

4. Let Q be a qo, for ℓ, ℓ' in Q set $\ell \prec \ell'$ if $\operatorname{dom} \ell = K_\ell \subseteq K_{\ell'} = \operatorname{dom} \ell'$, $rk_H(K_\ell) < rk_H(K_{\ell'})$ and $\ell' \restriction K_\ell = \ell$. It is a partial ranking of Q^{SCAT}, so take $f : [X_0]^{\mathbb{N}} \longrightarrow Q^{SCAT}$ a minimal bad weak Borel multiseq.

For all $X \in [X_0]^{\mathbb{N}}$ note $\ell_X = f(X) : K_X \longrightarrow Q$.

Hausdorff's theorem gives that for all $X \in [X_0]^{\mathbb{N}}$ either

(1) K_X is an α_X-sum, (2) K_X is an α_X^*-sum

(3) K_X is a singleton. (α^* is the reverse of α)

By GP there is $X_1 \in [X_0]^{\mathbb{N}}$ and there is $i \in \{1,2,3\}$ st $\forall X \in [X_1]^{\mathbb{N}}$ K_X is in (i). Case $i = 3$ yields a bad weak Borel Q-multisequence, as wanted.

We want to prove that neither $i = 1$ nor $i = 2$ are possible.

So suppose $i = 1$, the other case is similar.

Write $K_X = \sum_{\beta \in \alpha_X} L^X_\beta$, and call $s_X = (\ell_X \restriction L^X_\beta)_{\beta \in \alpha_X}$

Note that if

$$s_X \leq_{(Q^{SCAT})^{<ON}} s_{X^+} \text{ through } \iota_X : \alpha_X \longrightarrow \alpha_{X^+}$$

then $\forall \beta < \alpha_X$ there is $\sigma_\beta : L^X_\beta \longrightarrow L^{X^+}_{\iota_X(\beta)}$ st $\forall k \in L^X_\beta$ $\ell_X(k) \leq_Q \ell_{X^+}(\sigma_\beta(k))$, so $\sigma = \sum_{\beta \in \alpha_X} \sigma_\beta$ would witness that $f(X) \leq_{Q^{SCAT}} f(X^+)$, impossible!

So $X \longmapsto s_X$ is a bad weak Borel $(Q^{SCAT})^{<ON}$ multiseq. Since ON reflects bad multisequences, there is $X_2 \in [X_1]^{\mathbb{N}}$ st $\forall X \in [X_2]^{\mathbb{N}}$ there is

$\beta_x < \alpha_x$ such that $g : [X_2]^{\mathbb{N}} \longrightarrow \mathrm{SCAT}$ is

$$X \longmapsto \ell_x \upharpoonright L^x_{\beta_x}$$

a bad weak Borel Q^{SCAT}-multiseq.

Note finally that for all $X \in [X_2]^{\mathbb{N}}$ $g(X) <' f(X)$ thus contradicting the minimality of f. $\square$