LECTURE ROSENTHAL

RECALL - NOTATION

$$\ell_1 = \{ (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \mid \sum_{n=1}^{\infty} |x_n| < \infty \}$$

$$x \in \ell_1, \quad \|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

$$\ell_\infty = \{ (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \mid \sup_{n\in\mathbb{N}} |x_n| < \infty \}$$

$$x \in \ell_\infty, \quad \|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| < \infty$$

DUAL SPACE

$$\ell_1^* = \{ f : \ell_1 \to \mathbb{R} \mid f \text{ is continuous linear functional} \}$$

ℓ_1^* BANACH SPACE WITH

$$\|f\|_{\ell_1^*} = \sup_{\substack{\|x\| \le 1 \\ x \in \ell_1}} |f(x)|$$

NOTICE

$$\ell_1^* \cong \ell_\infty$$

$$\ell_\infty^*$$

NOTICE

There is an isometric isomorphism

$$C : B \longrightarrow B^{**} \qquad B = \text{Banach space}$$

$$x \longmapsto \underbrace{x(f)}_{\substack{\text{definition} \\ \text{element in bidual}}} = f(x)$$

WEAK CONVERGENCE

Let B be a Banach space and $(x_n)_{n\in\mathbb{N}} \subseteq B$.
We say that $(x_n)_{n\in\mathbb{N}}$ weak converge if

$$\underbrace{(f(x_n))_{n\in\mathbb{N}}}_{\subseteq\mathbb{R}} \longrightarrow \underbrace{f(x)}_{\in\mathbb{R}}$$

for each continuous linear functional f.

WEAKLY-CAUCHY IF $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy.

CONVERGENCE $\Rightarrow$ WEAK CONVERGENCE
$\not\Leftarrow$

EQUIVALENT TO THE USUAL ℓ^1-BASIS

Let $(x_n)_{n\in\omega}$ be a sequence in B Banach space.
We say that $(x_n)_{n\in\omega}$ is equivalent to the usual ℓ^1-basis if $\exists\, a, b \in \mathbb{R}^+$ such that $\forall n \in \mathbb{N}$ $\forall\, c_0, \dots, c_{n-1} \in \mathbb{R}$

$$a \sum_{i=0}^{n-1} |c_i| \leq \left\| \sum_{i=0}^{n-1} c_i x_i \right\| \leq b \sum_{i=0}^{n-1} |c_i|$$

NOTICE

This means $\overline{\operatorname{span}((x_n)_{n\in\omega})} \cong \ell_1$

$$f: \ell_1 \longrightarrow B$$

$$(c_n)_{n\in\omega} \longmapsto \sum_{i\in\mathbb{N}} c_i x_i$$

NOTATION

Let S be a set we define

$$\ell^\infty(S) = \{ f : S \longrightarrow \mathbb{R} \mid \|f\|_\infty := \sup_{n\in\omega} |f(x)| < \infty \}$$

ROSENTHAL THEOREM (1974)

Let S be a set and $(f_n)_{n\in\omega}$ be a bounded sequence in $\ell^\infty(S)$ then there exists $(f_{n_k})_{k\in\omega}$ such that

① (f_{n_k}) is weakly-Cauchy
[pointwise converge]

or

② (f_{n_k}) is equivalent to the usual ℓ^1-basis.

Why this theorem gives us information about the Banach spaces that have an isomorphic copy to ℓ^1?

COROLLARY

If B is a Banach space, then the following are equivalent:

① Every bounded sequence $(x_n)_{n\in\omega}$ in B has a weakly Cauchy subsequence

② ℓ^1 does not embed in X

PROOF

$\boxed{1 \Rightarrow 2}$

It is enough prove that the canonical basis $(e_n)_{n\in\omega}$ of ℓ^1 has not weakly Cauchy subsequence.

Let $(e_{n_k})_{k\in\omega}$ be a subsequence, then we define $f \in \ell^{1*} \cong \ell^\infty$ as follows

$$f(k) = \begin{cases} 1 & \text{if } \exists i\in\omega\,(k = n_{2i}) \\ 0 & \text{otherwis} \end{cases}$$

This $f \in \ell^\infty \cong \ell^{1*}$, so

$$f(e_{n_k}) = \sum_{i\in\omega} e_{n_k}(i) f(i) =$$

$$\underset{\substack{\uparrow \\ e_{n_k}(i)=0 \ \forall i\neq n_k \\ e_{n_k}(n_k)=1}}{=} f(n_k)$$

So

$$\left.\begin{array}{l} f(e_{n_0}) = 1 \\ f(e_{n_1}) = 0 \\ f(e_{n_2}) = 1 \\ \vdots \end{array}\right\} \begin{array}{l} (f(e_{n_k}))_{k\in\omega} \text{ is} \\ \text{not convergent} \Rightarrow \\ \Rightarrow (e_{n_k})_{k\in\omega} \text{ is not} \\ \text{weakly Cauchy} \end{array}$$

$\boxed{2 \Rightarrow 1}$

Since $X \hookrightarrow X^{**}$, then let $(x_n)_{n\in\omega} \subseteq X$ bounded, so we can see $(x_n)_{n\in\omega} \subseteq X^{**}$ bounded. Then we can see X as a functional in X^{**}, so is seen as

$$x : B_1(X^*) \longrightarrow \mathbb{R}$$
$$x(f) = f(x)$$

Then $\|x\|_\infty = \|f\| \, \|x\| = \|x\|$.

So, let us fix $S = B_1(X^*)$, then by Rosenthal we have that $(x_n)_{n\in\omega}$ admit a weakly Cauchy subsequence or an equivalent to the usual ℓ^1-basis subsequence.

Since X and $c(X) \subseteq X^{**}$ are isometric isomorphic, if $\ell^1 \not\hookrightarrow X$ then $\ell^1 \not\hookrightarrow c(X)$ and so 2 in Rosenthal is false. Then every $(x_n)_{n\in\omega}$ has a weakly Cauchy subsequence.

CODING WEAK CAUCHY CONVERGENCE AND EQUIVALENCE TO THE USUAL ℓ^1-BASIS IN COMBINATORICS

DEF 1

Let S be a set, $A, B \subseteq S$. We say that (A,B) is disjoint if $A \cap B = \emptyset$. Let $A_n, B_n \subseteq S$ we say that a sequence of disjoint pairs $((A_n, B_n))_{n\in\omega}$ is indipendent if

$\forall F, G \subseteq \omega$, F and G finite subset, then

$$\left(\bigcap_{n\in F} A_n\right) \cap \left(\bigcap_{n\in G} B_n\right) = \emptyset$$

RECALL

In the Rosenthal theorem we fixed

1. S a set
2. $(f_n)_{n\in\omega}$, where $f_n : S \to \mathbb{R}$
3. $(f_n)_{n\in\omega}$ is bounded, $\exists b \in \mathbb{R}^+$ s.t. $\forall n\in\omega (\|f_n\|_\infty < b)$

LEMMA 1

Let $r, s \in \mathbb{Q}$ and $r < s$. Let

$$A_n = A_n^{r,s} = \{ x \in S : f_n(x) < r \}$$

$$B_n = B_n^{r,s} = \{ x \in S : f_n(x) > s \}$$

If $((A_n, B_n)_{n \in \omega})$ is indipendent then $(f_n)_{n \in \omega}$ is equivalent to the usual ℓ^1-basis.

PROOF

Since $(f_n)_{n \in \omega}$ is bounded then $\forall n \in \omega$ $\forall c_0, \dots, c_{n-1} \in \mathbb{R}$

$$\left\| \sum_{i=0}^{n-1} c_i f_i \right\| \leq \sum_{i=0}^{n-1} |c_i| \, b = b \sum_{i=0}^{n-1} |c_i|$$

Now we want to find $a \in \mathbb{R}^+$: guess $a = \frac{s-r}{2}$

Let $F = \{ i \in n \mid c_i \geq 0 \}$, $G = \{ i \in n \mid c_i < 0 \}$

From the indipendence hypothesis $\exists x, y \in S$ such that

$$x \in \bigcap_{i \in F} A_i \cap \bigcap_{i \in G} B_i$$

$$y \in \bigcap_{i \in G} A_i \cap \bigcap_{i \in F} B_i$$

Let us define

$$c := \sum_{i \in M} c_i f_i(y) \geq \sum_{i \in F} |c_i| s - \sum_{i \in G} |c_i| r$$

$$d := \sum_{i \in M} c_i f_i(x) \leq \sum_{i \in F} |c_i| r - \sum_{i \in G} |c_i| s$$

$$\frac{(s-r)}{2} \sum_{i \in M} |c_i| \leq c - d \leq$$

$$\leq \sum_{i \in M} |c_i| \left(f_i(y) - f_i(x)\right) \leq$$

$$\leq \sum_{i \in M} |c_i| \, \|f_i\|_\infty = \Big\| \sum_{i \in M} c_i f_i \Big\|_\infty$$

$\square$

CONVERGENCE ON PAIRS OF SETS

Let S be a set and $((A_n, B_n))_{n \in \omega}$ a sequence of disjoint pairs. Let $X \subseteq S$.
We say that $((A_n, B_n))_{n \in \omega}$ converge on X if $\forall x \in X$ we have

for all but finitely many $n \in \omega$, $x \notin A_n$

or

for all but finitely many $n \in \omega$, $x \notin B_n$

If $X=S$ we say directly $((A_n, B_n))_{n\in\omega}$ converge.

LEMMA 2

If $\forall\, r,s\in\mathbb{Q}$, $r<s$, $((A_n^{r,s}, B_n^{r,s}))_{n\in\omega}$ is convergent then $(f_n)_{n\in\omega}$ is pointwise convergent.

PROOF

Let us suppose $\exists\, x\in S$ such that

$$\liminf f_n(x) < \limsup f_n(x)$$

Then there exists $r<s\in\mathbb{Q}$ such that

$$\liminf f_n(x) < r < s < \limsup f_n(x)$$

x belongs to infinitely many $A_n^{r,s}$ and infinitely many $B_n^{r,s}$, a contradiction. $\square$

LEMMA 3 (THIS STATEMENT IS ONLY COMBINATOR.)

Every sequence $((A_m, B_m))_{m \in \omega}$ of disjoint pairs has a convergent subsequence or an indipendent one.

PROOF

Let $P \subseteq [\mathbb{N}]^\omega$ be the set such that

$$(m_i)_{i \in \omega} \in P \iff \forall k \left[\bigcap_{\substack{i<k \\ i \text{ even}}} A_{m_i} \cap \bigcap_{\substack{i<k \\ i \text{ odd}}} B_{m_i} \neq \emptyset \right]$$

P is a closed subset, indeed if we write

$$P = \bigcap_{k \in \omega} P_k$$

where $P_k = \{ (m_i)_{i \in \omega} : \bigcap_{\substack{i<k \\ i \text{ even}}} A_{m_i} \cap \bigcap_{\substack{i<k \\ i \text{ odd}}} B_{m_i} \neq \emptyset \}$

P_k is a clopen condition, so P is closed.

Since P is closed, by GP there exists $H \in [\mathbb{N}]^\omega$ such that $\underbrace{[H]^\omega \subseteq P}_{\text{CASE 1}}$ or

$\underbrace{[H]^\omega \cap P = \emptyset}_{\text{CASE 2}}$

$\boxed{\text{CASE 1}}$

$[H]^{\omega} \subseteq P$, given $(m_i)_{i\in\omega}$ an increasing enumeration of H then $((A_{m_{2i+1}}, B_{2m_{i+1}}))_{m_i \in H}$

is indipendent. Indeed, let

$F, G \subseteq \omega$ such that $F \cap G = \emptyset$, finite sets,

WLOG $F \cup G = \{0, \dots, k-1\}$

$F, G \subseteq \mathbb{N}$ $\qquad \underbrace{F \cap G} = \emptyset$

$$\bigcap_{i\in F} A_{m_{2i+1}} \cap \bigcap_{i\in G} B_{m_{2i+1}} \neq \emptyset$$

Then, the $\exists\, \ell \geq k$ and $I = \{ m_i \mid i \in k \} \subseteq H$, with $m_i < m_{i+1}$ such that

$$\emptyset \neq \bigcap_{\substack{i<\ell \\ i \text{ even}}} A_{m_i} \cap \bigcap_{\substack{i<\ell \\ i \text{ odd}}} B_{m_i} \subseteq \bigcap_{i\in F} A_{m_{2i+1}} \cap \bigcap_{i\in G} B_{2i+1}$$

DEF

and so it is an indipendent sequence

$(\Rightarrow (f_n)_{n\in\omega}$ admit a subsequence equivalent to the usual ℓ^2-basis)

$\boxed{\text{CASE 2}}$

Let us suppose that $[H]^\omega \cap P = \emptyset$ and that $((A_m, B_m))_{m\in\omega}$ is not convergent.

Then, by definition, $\exists\, x \in X$ such that

$$I = \{m_i : x \in A_{m_i}\} \text{ is infinite}$$

$$J = \{m_i : x \in B_{m_i}\} \text{ is infinite}$$

By hypothesis $((A_n, B_n))_{n\in\omega}$ is disjoint then $I \cap J = \emptyset$, so there exists $\{m_i \mid i \in \omega\} \subseteq H$, with $m_i < m_{i+1}$ such that $\{m_i \mid i \text{ even}\} \subseteq I$ and $\{m_i \mid i \text{ odd}\} \subseteq J$, so $\forall k \in \omega$

$$\bigcap_{\substack{i<k \\ i \text{ even}}} A_{m_i} \cap \bigcap_{\substack{i<k \\ i \text{ odd}}} B_{m_i} \neq \emptyset$$

$\Rightarrow \{m_i \mid i \in \omega\} \in P \Rightarrow [H]^\omega \cap P \neq \emptyset$, contradiction.

ROSENTHAL THEOREM (1974)

Let S be a set and $(f_n)_{n\in\omega}$ be a bounded sequence in $\ell^\infty(S)$ then there exists $(f_{n_k})_{k\in\omega}$ such that

1. (f_{n_k}) is weakly-Cauchy

2. (f_{n_k}) is equivalent to the usual ℓ^1-basis.

NOTICE

If $((A_m, B_m))_{m\in\omega}$ is convergent then for each infinite subset $A \subseteq X_m$ follows $((A_m, B_m))_{m\in A}$ is convergent.

APPLICATION TO PROVE ROSENTHAL THEOREM

Let $(t_i)_{i\in\omega}$ an enumeration of $\{(r,s)\in\mathbb{Q}^2 \mid r<s\}$.

$\boxed{\text{STEP } 0}$

$r_{z_0} < s_{z_0} \Rightarrow ((A_n^{r_{z_0}, s_{z_0}}, B_n^{r_{z_0}, s_{z_0}}))_{n\in\omega}$ is disjoint. By LEMMA 3 we obtain $X_0 \subseteq \omega$ infinite subset such that one of the following hold

① $((A_n^{r_{z_0}, s_{z_0}}, B_n^{r_{z_0}, s_{z_0}}))_{n\in X_0}$ is independent, then by LEMMA 1 we have $(f_n)_{n\in X_0}$ is $\ell_1 B$;

② $((A_n^{r_{z_0}, s_{z_0}}, B_n^{r_{z_0}, s_{z_0}}))_{n\in X_0}$ is convergent, then we proceede with STEP $n+1$.

$\boxed{\text{STEP } n+1}$

Let us consider $(f_m)_{m\in X_n}$ and

$$((A_m^{r_{z_{n+1}}, s_{z_{n+1}}}, B_m^{r_{z_{n+1}}, s_{z_{n+1}}}))_{m\in X_n}.$$

From LEMMA 3 we obtain $X_{n+1} \subseteq X_n$ infinite subset such that

① $(f_m)_{m \in X_{n+1}} \in \ell_1 B \to$ STOP

② $\left(\left(A_m^{r_{z_{n+1}}, s_{z_{n+1}}}, B_m^{r_{z_{n+1}}, z_{n+1}}\right)\right)_{m \in X_{n+1}}$ is convergent

Now, let us suppose that we are always in CASE 2.

We have

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots \supseteq X_n \supseteq \dots$$

Now let us define X such that

$$x_0 := \min X_0 \in X$$

Let us suppose that $x_n \in X$ then

$$x_{n+1} := \min \{ y \in X_{n+1} \mid y > x_n \} \in X$$

So, $X \subseteq^* X_n$ for each $n \in X$, that is $\{ y \in X \mid y \geq x_n \} \subseteq X_n$

we obtain that for each $r < s \in \mathbb{Q}$

$$\left(\left(A_n^{r,s}, B_n^{r,s}\right)\right)_{n\in X} \text{ is convergent}$$

then $(f_n)_{n\in X}$ is weakly Cauchy

LEMMA 2