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LECTURE 7

31/5/2022

PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

Lemma Assume $\tau \supseteq \in_{\Delta_0}$ and $T \supseteq ZFC_\tau$ (1)

and $(T_{\forall\vee\exists} + \neg CH + ZFC_\tau)$ is consistent.

Then $CH \notin SCH(T)$ ~~consequently~~ equivalently it fails in some T-ec model.

Pf.: Let $\mathcal{M} \models ZFC_\tau + \neg CH + T_{\forall\vee\exists}$ $Th(\mathcal{M}) = R$

$SCH(T)_{\forall\vee\exists} = T_{\forall\vee\exists}$

$\{\varphi : \varphi \text{ is } \Pi_2 \text{ for } \tau \text{ and } \varphi + R_{\forall\vee\exists} \text{ is cons whenever } R \supseteq T_{\forall\vee\exists}\}$

$\mathcal{M} = \mathcal{M}_0 \sqsubseteq \mathcal{M}_1 \sqsubseteq \mathcal{M}_2 \sqsubseteq \mathcal{M}_3 \sqsubseteq \cdots \quad \mathcal{M}_\omega$

$\mathcal{M}_1 \models SCH(T) + R_{\forall\vee\exists}$, $\mathcal{M}_2 \models R$, $\mathcal{M}_3 \models SCH(T) + R_{\forall\vee\exists}$, $\mathcal{M}_\omega \models SCH(T) + \neg CH$

Rmk ZFC_{Δ_0} does not have a model companion

⑦

$$(H_{\omega_1}, \in_{\Delta_0}) \prec_1 (V, \in_{\Delta_0}) \models ZFC_{\Delta_0}$$

$$\Downarrow$$

$$\forall x \exists f (f : \omega \to x \text{ is surjective}) \in SCH(ZFC_{\Delta_0})$$
$$\cap\!\!\mid$$
$$KH(ZFC_{\Delta_0})$$

There exists for each n a Σ^1_{n+1} universal set

Σ^1_{n+1} on 2^ω corresponds to Σ_{n+1} on H_{ω_1}
which gives that there are Σ_2-def. subsets of H_{ω_1}
which are not Σ_1-definable

Fact Assume $\tau \supseteq \in_{\Delta_0} \cup \{\kappa\}$

$T \supseteq ZFC_\tau + \kappa$ is a cardinal and

$\forall x \exists F (F : \kappa \to x)$ is surj. belongs to $SCH(T)$

$(H_{\kappa^+}^{\mathcal{M}}, \tau^{\mathcal{M}}) \prec_1 \mathcal{M}$ whenever $\mathcal{M} \models T$.

Example: $\tau = \in_{\Delta_0}$ ~~$\cup \{\omega_2\}$~~ $T = ZFC_{\Delta_0}$ ~~$+ \omega_2$~~

$(H_{\omega_2}^{\mathcal{M}}, \in_{\Delta_0}^{\mathcal{M}}) \prec_1 \mathcal{M} \models ZFC_{\Delta_0}$ ϕ $\in_{\Delta_0}$-formula

$\tau = \in_{\Delta_0} \cup \{\omega_2\} \cup \{$ ~~A_ϕ~~ $, R_\phi$: $T \models$ ~~$\forall x (\phi(x) \to x \subseteq \omega_2)$~~

B ″ $T \models \forall x_1 \cdots x_n \left(\phi(x_1 \cdots x_n) \to \bigwedge_{i=1}^{n} x_i \subseteq \omega_2\right)$

if $T \supseteq ZFC_{\Delta_0} + \omega_2$ is the first unc. card.

$+ \forall x_1 \cdots x_n \left(R_\phi(x_1 \cdots x_n) \leftrightarrow \phi(x_1 \cdots x_n)\right)$

$$\left(H^{n}_{\omega_2}, \tau^{n}\right) \prec_1 \mathfrak{E}^{n} \models T$$

<u>Goal</u>

Make the theory of H_{κ^+} model complete.

<u>Def</u>: Given $a \in H_{\kappa^+}$, $R \subseteq P(\kappa)^2$ codes a if:

- R is a well founded extensional relation on $\alpha \leq \kappa$
- As a directed graph 0 is the top element of R

$(\alpha, R) \models \mathrm{Ext}$ and R is w.founded on α.

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$trcl\{\emptyset, \{\emptyset\}\}$

$trcl(\{\{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$

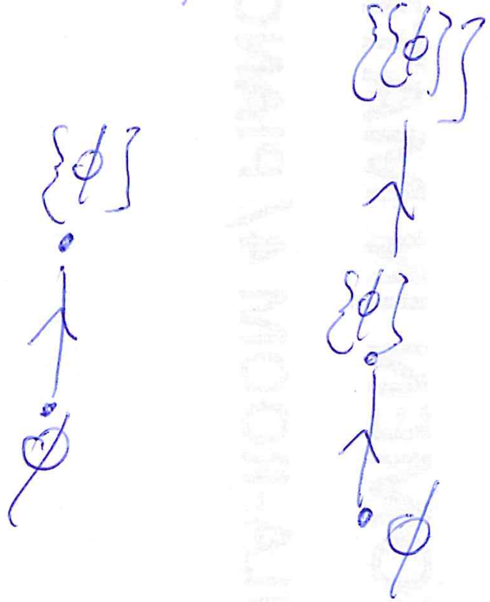
$\{\{\emptyset\}\}$

$tc(\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\{\{\emptyset\}\}\}\}\})$

$U^0\{a\} = a$ $\quad U^{n+1}(a) = U(U^n(a))$

$U^0\{a\} = \{a\}$

$trcl(\{a\}) = \bigcup_{n \in \omega} U^n a$



$trcl(\{\emptyset\}) = \{\{\emptyset\}, \emptyset\}$

$trcl(\{\{\emptyset\}\}) = \{\{\{\emptyset\}\}, \{\emptyset\}, \emptyset\}$

Let $\Pi_R : \alpha \longrightarrow H_{\kappa^+}$

$\beta \longmapsto \{\Pi_R(\xi) : \xi R \beta\}$

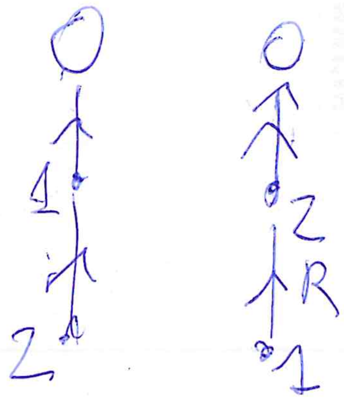
$d = \Pi_R(0)$

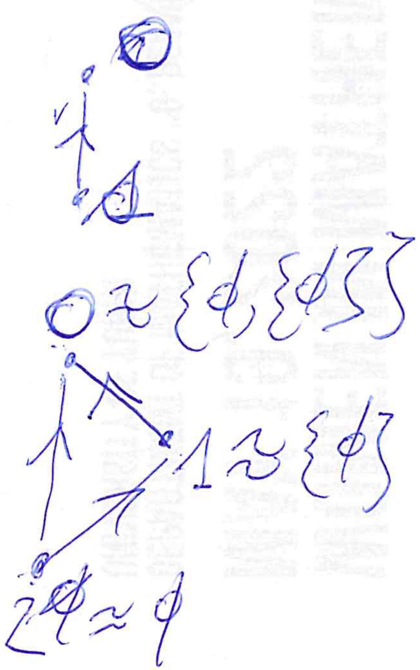
ϕ

tcl $\{\phi\}$

$\{\phi, \{\phi\}\}$

$\{\{\phi\}\}$





Def: $WFE_K = \{R \subseteq K^2 :$ K is well founded extensional & 0 is its top element$\}$

$H_{K^+} = \{x : |tc(x)| < K^+\}$

$a = Coll_K(R)$ if

$\Pi_R : dom(R) \to H_{K^+}$

$0 \mapsto a$

$\Pi_R[dom(R)] = tc(\{\Pi_R(0)\})$

f is a function ran(f) ⊆ dom(R)

$\forall f \left(dom(f) = \omega \to \exists n \in \omega \neg f(n+1) R f(n) \right)$

$\forall z \forall p \in dom(R) \left(z = p \Leftrightarrow \forall y \in dom(R) (y R z \Leftrightarrow y R p) \right)$

$a \in b$ iff R codes a

S codes b

$$E = \{ (R,S) \in WFE_K^2 : \exists f : dom(R) \to dom(S) \text{ with } f(0)\, S\, 0 \text{ and } f \text{ is an isomorphism of } (dom(R), R) \text{ with } (Ind_S(f(0)), S)$$

$$= \{ i : \exists \alpha_1 \ldots \alpha_k \; i\, S\, \alpha_1\, S\, \alpha_2\, S \ldots S\, \alpha_k\, S\, f(0$$

Fact

$$(WFE/\sim, E/\sim) \cong (H_{\kappa^+}, \in)$$

$R \sim S$ iff $\exists f : dom(R) \to dom(S)$ isomorphism of $(dom(R), R)$ with $(dom(S), S)$ and $f(0)=0$

$[S]_\sim \; E/\sim \; [R]$ is well defined

Example of $\tau \supseteq \in_{\Delta_0} \cup \{K\}$ s.t.

~~$Th(H_{k^+}$~~ $\{\psi : \forall \mathcal{M} \models ZFC_\tau \ (H_{k^+}^{\mathcal{M}}, \tau^{\mathcal{M}}) \models \psi\}$

~~ψ is~~ + K is an infinite cardinal

↑

this is model complete

$\tau = \in_{\Delta_0} \cup \{K\} \cup \{R_\phi : ZFC_{\Delta_0} \models \forall x_1 \dots x_n \left(R_\phi(x_1 \dots x_n) \to \bigwedge_{i=1}^{n} x_i \text{ cl}\right.$

$ZFC_\tau = ZFC_{\Delta_0}$ + K is an infinite cardinal +

$\forall x_1 \dots x_n \left(\phi(x_1 \dots x_n) \longleftrightarrow R_\phi(x_1 \dots x_n)\right)$

(12)

$$R^{H_k+}_{x=y} = \{(X,Y) \in WFE^2_k : Cod_k(X) = Cod_k(Y)\}$$

$$R^{H_k+}_{x\in y} = \{(X,Y) \in WFE^2_k : Cod_k(X) \in Cod_k(Y)\}$$

$$R^{H_k+}_{\phi\wedge\psi} = \{(X_1 \ldots X_n) \in WFE^n_k : R^{H_k+}_{\phi}(X_1 \ldots X_n) \wedge R^{H_k+}_{\psi}(X_1 \ldots X_n)$$

⋮

$R_{\phi\vee\psi}$

$R_{\neg\phi}$

$$R_{\exists x\phi(x,x_2..x_n)} = \{(X_1 \ldots X_n) \in WFE^n_k : \exists X \in WFE^{1}_k \; R_{\phi}(X, X_2 \ldots X_n)\}$$

$$\left(WFE/_{\sim}, E/_{\sim}, = \right) \models \psi([X_1] \dots [X_n])$$ (13)

ψ $\in$-formula $Cod_k(X_i) = a_i$ for $i = 1 \dots n$

$$(H_{k^+}, \tau) \models \forall X_1 \dots X_n \left(\bigwedge_{i=1}^{n} WFE_k(X_i) \land \right.$$

$WFE_k(X_i)$: atomic for τ

$$\bigwedge_{i=1}^{n} Cod_k(X_i) = a_i$$

$$\downarrow$$

$$\left. \psi(a_1 \dots a_n) \right)$$

for φ $\in$-formula and $a_1 \dots a_m \in H_{\kappa^+}$

(14)

$$(H_{\kappa^+}, \in) \models \varphi(a_1 \dots a_m)$$

$$\Updownarrow$$

$$(H_{\kappa^+}, \tau) \models \forall X_1 \dots X_m \Big[\Big(\bigwedge_{i=1}^{m} \underbrace{WFE_\kappa(X_i)}_{\text{atomic}} \wedge \overbrace{Cod_\kappa(X_i) = a_i}^{\Pi_1} \Big) \leftrightarrow \varphi(a_1 \dots a_m) \Big]$$

means that ~~H~~ $Th(H_{\kappa^+})$ is model complete

$Cod_k(x) = y$ is Π_1 in $\in_{\Delta_0} \cup \{k\}$ (15)

~~$\emptyset$~~ $x \in$ ~~W~~WFE_k ~~$(x$~~

Π_1 in $\in_{\Delta_0}$ or atomic in τ

or Σ_1

$\Pi_x(0) = y$ iff $\exists F$ ($dom(F) = dom(x) \wedge$

$ran(F)$ is transitive $\wedge$

$\forall z, w \in dom(x) (z R w \leftrightarrow F(z) \in F(w))$

$\wedge F(0) = y$)