PHD COURSE
ON
MODEL COMPANIONSHIP RESULTS
FOR
SET THEORY

LECTURE 9 15/6/2022

Recall ①

$\in_{\Delta_1} = \in \cup \{ R_\varphi, f_\psi : \varphi, \psi \text{ are } \Delta_1(ZFC^-),\ \psi\ ZFC^- \models \forall \vec{x}\, \exists! y\, \psi(\vec{x}, y) \}$

$NS_{\omega_1} = \{ X \subseteq \omega_1 : \exists C \subseteq \omega_1 \text{ club } C \cap X = \emptyset \}$

~~NS~~

<u>Thm</u>. Let $(V, \in) \models ZFC + LC$ ← (exists class many Woodins) and G be V-generic for some forcing $P \in V$. Then

$$(V, \in_{\Delta_1}, \omega_1^V, NS^V) \equiv_1 (V[G], \omega_1^{V[G]}, NS^{V[G]}, \in_{\Delta_1}^{V[G]})$$

$\not\prec_1$ if $P = Coll(\omega, \omega_1^V)$

Remark: consider

$$\exists x \left[(x = \omega_1^L) \wedge x = \omega_1 \right] \qquad \omega_1 = \omega_1^L$$

in $\in_{\Delta_0} \cup \{\omega_1\}$ ~~in~~ this is an existential sentence
or

if G is L-generic for $\mathrm{Coll}(\omega, \omega_1^L)$

$$(L, \in_{\Delta_0}, \omega_1^L) \models \omega_1 = \omega_1^L \qquad (L[G], \in_{\Delta_0}, \omega_1^{L[G]}) \not\models \omega_1 = \omega_1^L$$

<u>Lemma</u> (Fodor) NS is closed under diagonal unions.

<u>Def</u>: Given $\{X_\alpha : \alpha < \omega_1\} \subseteq P(\omega_1)$

$$\nabla_{\alpha<\omega_1} X_\alpha = \{\beta : \exists \alpha < \beta \ \beta \in X_\alpha\}$$

$$\Delta_{\alpha<\omega_1} X_\alpha = \{\beta : \forall \alpha < \beta \ \beta \in X_\alpha\}$$

Fact: Given $f : \omega_1 \to \omega_2$

$C_f = \{\alpha < \omega_2 : f[\alpha] \subseteq \alpha\}$ is a club.

Pf closed: easy if $\{\alpha_i : i \in I\} \subseteq C_f$

with $|I| \leq \aleph_0$ $\sup_{i \in I} \alpha_i = \alpha \in C_f$

if $\beta \in C_f$ $\exists i \in I$ $\beta \in \alpha_i \Rightarrow f(\beta) \in \alpha_i \subseteq \alpha$.

unb.: let $\beta \in \omega_1$ find $X \subseteq \omega_2$ s.t. $\beta \in X$

X is countable and ~~$(X, f \restriction X) \prec (\omega_1, f)$~~

if (ω_1, f, g) $g : \omega_1 \times \omega \to \omega_1$ $g \restriction \{\alpha\} \times \omega$ is a surjection on α

$(X, f \restriction X, g \restriction X) \prec (\omega_2, f, g) \Rightarrow X$ is a countable ordinal

$X = \beta \Rightarrow (\beta, f\upharpoonright\beta) < (\omega_1, f) \Rightarrow f[\beta] \subseteq \beta$ 5

<u>Pf of Fodor Lemma</u>

We show that $\breve{NS} = \{X^c : X \in NS\} =$ "club filter" $= CF$ is closed under diagonal intersections.

$$\left(\Delta_{\alpha<\omega_1} X_\alpha\right)^c = \nabla_{\alpha<\omega_1}\left(X_\alpha^c\right) \Leftarrow \left(\nabla_{\alpha<\omega_1} X_\alpha\right)^c = \Delta_{\alpha<\omega_1}\left(X_\alpha^c\right)$$

~~given $\{C_\alpha : \alpha<\omega_1\}$ family of clubs~~

~~define $f(\beta) = \min \bigcap_{\alpha<\beta} C_\alpha$ $f : \omega_1 \to \omega_1$~~

Let $\theta > 2^{|P(\omega_2)|}$ be regular (6)

and $M \prec (H_\theta, \in)$ with M-countable

$\{C_\alpha : \alpha < \omega_2\} \in M$

family of clubs.

$\boxed{\underline{\text{Claim}} \quad M \cap \omega_1 \in \underset{\alpha<\omega_1}{\Delta} C_\alpha}$

Let $C^* = \{M : M \prec (H_\theta, \in) \quad \{C_\alpha : \alpha < \omega_2\} \in M\}$

$C = \{M \cap \omega_1 : M \in C^*\} \supseteq$ club

(1) $\omega_2 \in M$ $\qquad\qquad$ ω_1 is def in $(H_\theta, \in)$ (7)

M countable $\Rightarrow$ $M \cap \omega_1$ is a countable set

for any $\alpha \in M \cap \omega_1$ $\quad (H_\theta, \in) \models \exists f : \omega \to \alpha$ surjection

$(M, \in) \models \exists f : \omega \to \alpha$ surjection

$\omega \subseteq M \qquad \phi \in M \qquad \{\phi\} \in M$ and $x \in M \Rightarrow x \cup \{x\} \in M$

$\Downarrow$

$\alpha = f[\omega] \subseteq M \Rightarrow \left(\alpha \in M \cap \omega_1 \Rightarrow \alpha \subseteq M \right) \Rightarrow M \cap \omega_1$ is an ordinal.

$M \cap \omega_2 = \delta_M \in \cancel{\triangle}_{\alpha<\omega_1} \{C_\alpha : \alpha < \omega_1\}$ ~~⌀~~

(8)

$\delta_M \in C_\beta \quad \forall \beta \in \delta_M$

$\beta \in \delta_M \Rightarrow \beta \in M \Rightarrow C_\beta \in M \Rightarrow \delta_M \in \lim C_\beta$

$\Rightarrow \delta_M \in C_\beta$

$M \models C_\beta$ is a club $\Rightarrow$

~~M~~ $\forall \alpha \in \delta_M \ \exists \beta \in \delta_M \cap C_\beta$

$C^* = \{M : M \in [H_\theta]^{\aleph_0} \ (M, \in) \prec (H_\theta, \in)\}$

$C' = \{M \cap \omega_2 : M \in C^*\}$ contains a club | Claim

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Pf: $M_0 \in C^*$

$M_0 \cap \omega_1$

$M_1 \supseteq M_0 \cup \{\delta_{M_0}\}$

$M_\alpha \cap \omega_1$

$\shortparallel$

$M_{\alpha+1} \supseteq M_\alpha \cup \{\delta_{M_\alpha}\}$

$M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for β limit

$\{M_\alpha : \alpha<\omega_1\} \subseteq C^*$ and

$\{\delta_{M_\alpha} : \alpha<\omega_1\}$ is a club

$\blacksquare$

Lemma; Let $S \in NS^+ = P(\omega_1) \setminus NS$

S is stationary.

$f : S \to \omega_1$ s.t. $f(\alpha) \in \alpha$ $\forall \alpha \in S$.

Then $\exists \beta \in \omega_2$ s.t $S_\beta = \{\alpha \in S : f(\alpha) = \beta\}$ is stat.

Pf Note that $S \cup S^c = \omega_1$ is a club.

We prove the Lemma in case S is stationary

$$\{S_\beta : \beta < \omega_1\} \subseteq NS \Rightarrow \nabla_{\alpha < \omega_2} S_\beta \in NS.$$

$$\nabla_{\alpha<\omega_1} S_\beta = \{\gamma : \exists \beta<\gamma \ \ \gamma \in S_\beta\} = \{\gamma : \exists \beta<\gamma \ \ f(\gamma)=\beta\} =$$

$= \{S$ stat. $\lightning$ 11

NS is an ideal on $P(\omega_1)$ which extends the Frechet ideal $[\omega_1]^{<\omega}$

ex (even the bdd)

so if G is a ultrafilter on $P(\omega_1)/NS$

$\{S : [S]_{NS} \in G\}$ is a non principal ultrafilter on ω_1

Hence if $NS \in H_\theta$ and $G \in St(P(\omega_1)/NS)$

~~one~~ $Ult(H_\theta, G)$

$$\begin{array}{c} \| \\ \{[f]_G : f : \omega_1 \to H_\theta\} \end{array}$$

Note if $\theta > \omega_1$ is regular

and $f : \omega_1 \to H_\theta \Rightarrow f \in H_\theta$

$\{f : f : \omega_1 \to H_\theta\} \subseteq H_\theta$.

Def: NS is saturated if any maximal antichain of $P(\omega_1)/NS$ has size at most $\aleph_1$ (14)

$\{[S]_{NS} : S \subseteq \omega_1\}$ $\quad [S]_{NS} = [T]_{NS}$ iff

$S \Delta T \in NS \iff \exists C$ club $S \cap C = T \cap C$

$A \subseteq P(\omega_1)/NS$ is a maximal ant.

$\parallel$

$\{[S]_\alpha : \alpha < \omega_1\} \Rightarrow \bigvee_{P(\omega_1)/NS} \{[S_\alpha] : \alpha < \omega_1\} = [\omega_1]$

$$\boxed{\underline{\text{Fact}}\quad \bigvee_{P(\omega_1)/NS} \{[S_\alpha] : \alpha<\omega_2\} = \underbrace{\Big[\nabla_{\alpha<\omega_1} S_\alpha\Big]}_{S}}$$

$$[S_\alpha] \leqslant \Big[\nabla_{\alpha<\omega_2} S_\alpha\Big]$$

$$\|$$

$$S_\alpha \setminus \alpha{+}1 \subseteq \nabla_{\beta<\omega_2} S_\beta = \{\beta : \exists\gamma<\beta \quad \beta\in S_\gamma\}$$

~~If~~ $[S]$ is not the eub of $\{[S_\alpha] : \alpha<\omega_2\}$

$\exists\ T \quad [T] \geqslant [S_\alpha]$ for all $\alpha<\omega_2$ but

$[T] \not\geqslant [S] \qquad [T]\wedge[S] = \underset{\substack{\wedge \\ [S]}}{[T\cap S]} \geqslant [S_\alpha]$ for $\alpha<\omega_2$

Let $U = S \setminus (T \cap S)$ $[\emptyset] < [U] \leq [S]$ 16

$\cancel{[U] \cap}$ $\underbrace{U \cap S_\alpha \in NS}$

U

$\cap$

$S = \{\beta \in \omega_1 : \exists \alpha < \beta \ \beta \in S_\alpha\}$

$\cup$

$\cancel{U = \{\beta \in \omega_1 :}$

$\forall \beta \in U \ \exists f(\beta) \in \beta$ s.t. $\beta \in S_{f(\beta)}$

by Fodor $\exists \cancel{\beta}\, \gamma$ s.t. $\{\beta \in U : f(\beta) = \gamma\}$ is stat

$\|$

$\{\beta \in U : \beta \in S_\gamma\} = U \cap S_\gamma$ ↯

Lemma Assume G is V-generic for (17)

$P(\omega_2)/NS$ and NS is saturated.

Then $\mathrm{Ult}(V,G)$ in $V[G]$ is well founded.

Pf: Assume not find $\{f_n : n\in\omega\}\in V[G]$

s.t. $[f_{n+1}]_G \in [f_n]_G$ $\forall n\in\omega$.

Let ~~$\tau\in V$~~ $B = P(\omega_2)/NS$ and $\tau\in V^B$

$\tau_G = \{f_n : n\in\omega\}$ $\boxed{b} = [\![\tau : \omega \to V$ is a function s.t $\forall n\, \tau(n) : \omega_1 \to \mathrm{Ord} \wedge [\tau(n+1)]_{\dot G} \in [\tau(n)]_{\dot G}]\!] > 0_B$

now for each n let 18

$$D_n = \left\{ [S] : [S] \Vdash \tau(n) = \check{h}_S \text{ for some } \begin{array}{l} h_S : \omega_1 \to V \\ h_S \in V \end{array} \right\}$$

D_n is dense in $P(\omega_1)/NS$ below $[S]$.

find $A_n \subseteq D_n$ max antichain on $P(\omega_1)/NS$

and let $A_n = \{[S_\alpha^n] : \alpha < \omega_1\}$

$S_\alpha^n \cap S_\beta^n \in NS \quad \forall \alpha \neq \beta < \omega_1$

$$\left[\nabla_{\alpha<\omega_1} S_\alpha^n\right] = \bigvee_{P(\omega_1)/NS} \{[S_\alpha^n] : \alpha < \omega_1\} = [S]$$

w.l.o.g we can assume that (19)

$S_\alpha^n \cap S_\beta^n = \emptyset \quad \forall \alpha \neq \beta \quad [S_\alpha^n] \in A_n$

$\nabla_{\alpha < \omega_2} S_\alpha^n$ contains a club in S

$h_n : S \to V$

$\alpha \mapsto 0 \quad \alpha \notin S$

$\alpha \mapsto h_{S_\alpha^n}(\alpha)$ if $\alpha \in S_\alpha^n$

$\{h_n : n \in \omega\} \in V \qquad [S] \Vdash \tau(n) = \check{h}_n$

$S_\alpha^n \Vdash \tau(n) = h_{S_\alpha^n}$

$S_\alpha^n \Vdash \tau(\check{n})(\check{\beta}) = \check{h}_{S_\alpha^n}(\check{\beta}) = \check{h}(\check{\beta}) \quad \forall \beta \in S_\alpha^n$

$\{h_n : n \in \omega\}$

$[S] \Vdash \left([\dot{h}_{n+1}]_{\dot{G}} \in [\dot{h}_n]_{\dot{G}} \ \forall n \in \check{\omega} \right)$

$\forall n \Big[\{\alpha < \omega_1 : h_{n+1}(\alpha) \in h_n(\alpha)\} \in G$ whenever

G is V-generic for $P(\omega_1)/NS$ s.t. $S \in G$

$[\{\alpha \in S : h_{n+1}(\alpha) \in h_n(\alpha)\}] = [S]$

So for each $\exists\, C_n$ club s.t.

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$$\{\alpha \in S : h_{n+1}(\alpha) \in h_n(\alpha)\} \supseteq C_n \cap S$$

$$\{\alpha \in S : \forall n\ h_{n+1}(\alpha) \in h_n(\alpha)\} \supseteq \underbrace{\left(\bigcap_{n\in\omega} C_n\right) \cap S}_{\neq \emptyset}$$

$\blacksquare$

Cor Assume

$(M,\in) \models ZFC + NS$ is saturated

M is transitive and countable.

Then $\exists G$ M-generic for $\left(\frac{P(\omega_1)}{NS}\right)^M$

and ~~in~~ $(M[G],\in) \models Ult(M,G)$ is well-founded

$(V,\in) \models Ult(M,G)$ is well-founded.

$\blacksquare$

Def. Let $(M,\in) \models ZFC + NS_{\omega_1}$ is saturated
M countable and transitive

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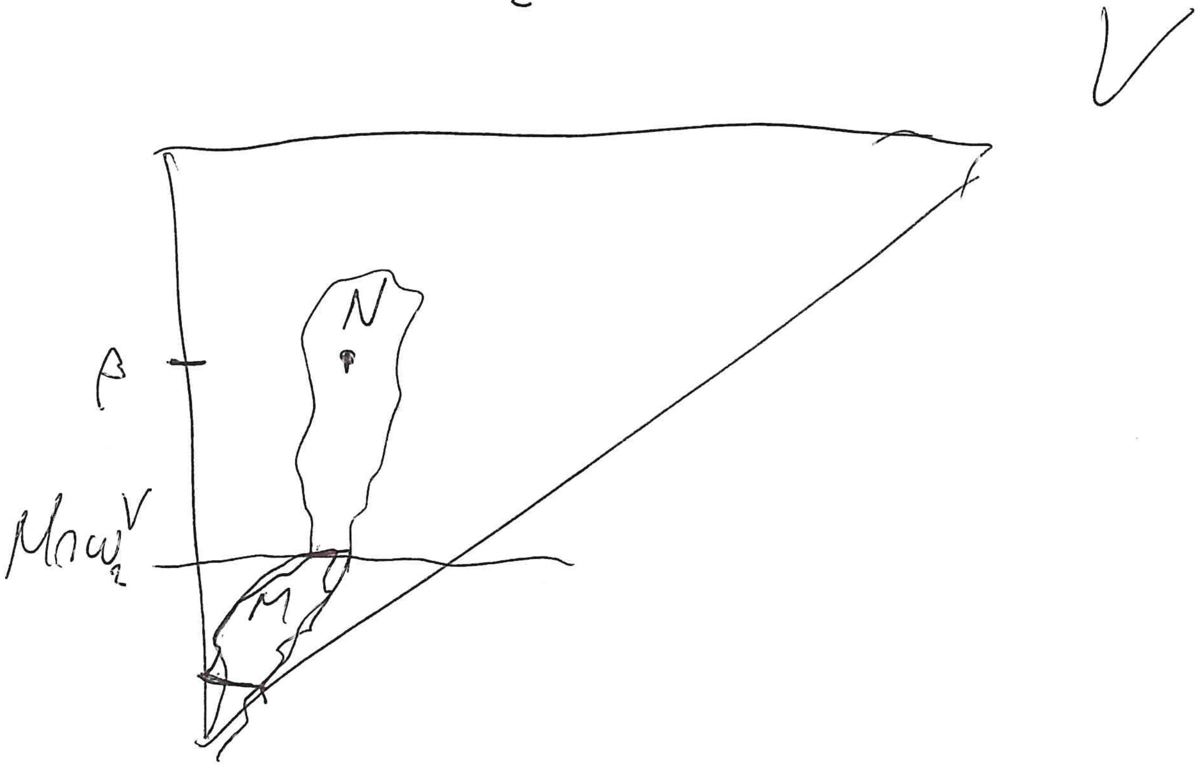
$(M,\in)$ is iterable if $\forall \alpha < \omega_1^V$

$\exists N \supseteq M$ s.t. $N \cap \omega_1^V \supseteq \alpha$ N is transitive

$M = (V_{M \cap \omega_1^V})^N$ $N \models ZFC + NS_{\omega_2}$ is saturated



<u>Lemma</u> If $(V, \in) \models NS_{\omega_1}$ is saturated and $(V, \in) \models \exists K$ K is meas.

Then $(V, \in) \models \exists$ an iterable M.