PHD COURSE ON MODEL COMPANIONSHIP RESULTS FOR SET THEORY

LECTURE 11 21/6/2022

$\omega \approx$ shaply of $P(X)$ for all finite ~~$n \in \omega$~~ sets X 1

$P(\omega) \approx$ second order arithmetic

$\downarrow$

$P(P(\omega)) \approx$ third order arithmetic H_κ V_α

$P(\kappa)$ Cod_κ $\approx$ $H_{\kappa^+} + \Sigma_1$ ✓

Cor. If $T \supseteq ZFC_\tau$ and $\tau \supseteq \in_{\Delta_0} \cup \{\omega_2\}$ (2)

and $T_{\forall\vee\exists} + \neg CH$ is consistent, then

$CH \notin SCH(T) \models Z^-_{\Delta_0}$ + certain amount + replacement

AC

$\{\Pi_2\text{-sent.}\ \psi$ for τ s.t. $\psi + R_{\forall\vee\exists}$ is cons. $\forall R \supseteq T_{\forall\vee\exists}\} = AMC(T)$ if AMC of T exists

$(+ \forall X \exists F\ F : \omega_2 \to X$ is surjection)

if there is a constant for ω_2

i.e. $\tau \ni \in_{\Delta_0} \cup \{\omega_2\}$

$T \models \omega_2$ is first inc. card.

$\cap$|

$KH(T) =$

$\{\Pi_2\text{-sent for } \tau\ \psi$ s.t. $M \models \psi$ iff M is T-ec$\} = MC(T)$ if $MC(T)$ exists

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- $\neg CH \in SCH(T)$ for any $T \supseteq ZFC_\tau + LC$

a theory of H_{ω_2}

(exists class many Woodin)

$$\in_{\Delta_0} \cup \{\omega_1\} \subseteq \tau \subseteq \in_{\Delta_1} \cup \{\omega_1, NS\} \cup \left(\text{reasonable families of UB-sets} \right)$$

More generally

- if $\in_{\Delta_0} \cup \{\omega_1\} \subseteq \tau$ and $T \supseteq ZFC_\tau + \omega_1$ is first unc. card.

and ψ is Π_2 $\neg\psi \notin SCH(T)$ if $\psi + T^{\tau}_{\forall\vee\exists}$ is cons.

- $\in_{\Delta_0} \cup \{\omega_1\} \subseteq \tau \subseteq \in_{\Delta_1} \cup \{\omega_1, NS\} \cup \{$certain UB-sets$\}$ and $T \supseteq ZFC_\tau + LC + \omega_1$ is first unc. card and ψ is Π_2 for τ

$$\psi \in SCH(T) \quad \text{iff} \quad T \models \exists P \; P \Vdash \psi^{H_{\omega_2}}$$

Moore (Aspero, Caicedo Velickovic, Todorcevic, Woodin)

There is a Π_2-sentence ψ_{Moore} in $\in_{\Delta_1} \cup \{NS, \omega_1\}$ τ_0 "

proper or semiproper or SSP

s.t.

$$ZFC \nvDash \exists P \; P \Vdash \psi_{Moore} \Rightarrow \psi_{Moore} \in SCH(ZFC_\tau)$$

for any $\tau \subseteq \in_{\Delta_1} \cup \{NS, \omega_1\}$ $\cup$ {certain UB-sets}

$\supseteq \in_{\Delta_1} \cup \{NS, \omega_1\}$

$ZFC_{\tau_0}^- + NS$ is the non-stat, ω_1 is first unc. card

$+ \psi \models$ there is a definable w.o. of $P(\omega)$ in type $\aleph_2$

(5)

~~∄~~ there is $\varphi(x, y, z, w)$ ~~atomic~~ formula of τ_0

s.t. T_0 $\boxed{ZFC^-_{\tau_0} + NS \text{ is stat} + \omega_2 \text{ is first unc. card} + b \text{ is something} + c \text{ is something}}$ + more

$\forall z \,(z \subseteq \omega \rightarrow \exists \alpha$ α is an ordinal of size at most $\aleph_2$ and $\varphi(z, \alpha, b, c)$

$T_0 \vdash \forall \alpha \; \forall z \; \forall s \; \big(z \subseteq \omega \wedge s \subseteq \omega \wedge \varphi(z, \alpha, b, c) \wedge \varphi(s, \alpha, b, c) \rightarrow z = s \big)$

⑥

~~$\mathcal{B}$~~ **Cor** if $T \supseteq ZFC_\tau$ and

$\tau \supseteq \in_{\Delta_1} \cup \{\omega_1, NS\}$ and

$T^\tau_{\forall\vee\exists} + \psi_{Moore}$ is consistent

$2^{\aleph_0} \cancel{>} \aleph_2 \notin SCH(T)$

and if $\in_{\Delta_0} \cup \{\omega_1, NS\}$ ($\cup$ certain UB-sets) $\supseteq \tau$

$SCH(T) \models$ there is a definable surjection of the ordinals of size at most ω_2 onto $P(\omega)$

$2^{\aleph_0} \overset{\psi}{=} \aleph_2$

$$\{V_\alpha[G] : G \text{ is } V\text{-generic for some } P \in V\}$$ 7

$$\|$$

$$\{V_\alpha : \alpha \in \mathrm{Ord}\}$$

- Generic multiverse over M ctm of ZFC

$$\{\cancel{M} V_\alpha^{M[G]} : G \ M\text{-generic for some } P \in M\}$$

Def.: Given B a ba

~~M is~~ a B-valued model for τ

~~iff~~: ~~is~~ $\mathcal{M} = (M, R^{\mathcal{M}} : R \in \tau$ rel-symbol, $f^{\mathcal{M}} : f \in \tau$ funct symbol, $c^{\mathcal{M}} : c \in \tau$ constant

- $(a_1 \dots a_n) \mapsto [\![R(a_1 \dots a_n)]\!]^{\mathcal{M}}_B$

- $R^{\mathcal{M}} : M^n \to B$ where n is the arity of R

- $f^{\mathcal{M}} : M^n \to M$ n is the arity of f

- $c^{\mathcal{M}} \in M$

- $=^{\mathcal{M}} : M^2 \to B$

  $(a,b) \mapsto [\![a = b]\!]^{\mathcal{M}}_B$

- $[\![\tau = \tau]\!] = 1_B \quad \forall\, \tau \in M$

- $[\![\tau = \sigma]\!] = [\![\sigma = \tau]\!] \quad \forall\, \tau, \sigma \in M$

- $[\![\tau = \sigma]\!] \wedge [\![\sigma = \rho]\!] \leq [\![\tau = \rho]\!] \quad \forall\, \tau, \sigma, \rho \in M$

- $[\![R(\tau_1 \ldots \tau_n)]\!] \wedge \bigwedge_{i=1}^{n} [\![\tau_i = \sigma_i]\!] \leq [\![R(\sigma_1 \ldots \sigma_n)]\!]$

- $[\![f(\tau_1 \ldots \tau_n) = \sigma]\!] \wedge \bigwedge_{i=1}^{n} [\![\tau_i = \sigma_i]\!] \leq [\![f(\sigma_1 \ldots \sigma_n) = \sigma]\!]$

given a τ-formula $\varphi(x_1 \ldots x_n)$ (10)

and v : Free Var $\longrightarrow M$

$$[\![\varphi]\!]_B^{M,v} = [\![R(v(t_1) \ldots v(t_n))]\!]_B^M \quad \text{if } \varphi \text{ is } \cancel{R(x_1 \ldots x_n)}$$

$$R(t_1(\vec{x}) \ldots t_n(\vec{x}))$$

$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \wedge [\![\psi]\!]$$

$$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \vee [\![\psi]\!]$$

$$[\![\neg \varphi]\!] = \neg_{RO(B)} [\![\varphi]\!]$$

$$[\![\exists x\, \varphi(x)]\!]_B^{M,v} = \bigvee_{RO(B)} \left\{ [\![\varphi]\!]_B^{M,v_\tau} \;:\; \text{with } \begin{array}{l} v_\tau(y) = v(y) \text{ for } y \neq x \\ v_\tau(x) = \tau \end{array} \right\}$$

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Def: Given $\mathcal{M}$ B-valued for τ
and $\mathcal{N}$ C-valued for τ

(ι, ϕ) is a boolean morphism of $\mathcal{M}$ ~~into~~ to $\mathcal{N}$ if

$\phi : M \to N$

$\iota : RO(B) \to RO(C)$ is a complete homomorphism s.t. $\iota \restriction B : B \to C$

$$\iota\left([\![\tau = \sigma]\!]^{\mathcal{M}}_{B}\right) \leq [\![\phi(\tau) = \phi(\sigma)]\!]^{\mathcal{N}}_{C}$$

$$\iota\left([\![R(\tau_1 \dots \tau_n)]\!]^{\mathcal{M}}_{B}\right) \leq [\![R(\phi(\tau_1) \dots \phi(\tau_n))]\!]^{\mathcal{N}}_{C}$$

$$\iota\left(\cancel{[\![\bigwedge_{i=1}^{n} \sigma_i = \tau_i]\!]^{\mathcal{M}}_{B}}\ [\![f(\sigma_1 \dots \sigma_n) = \sigma]\!]^{\mathcal{M}}_{B}\right) \leq [\![f(\phi(\sigma_1) \dots \phi(\sigma_n)) = \phi(\sigma)]\!]^{\mathcal{N}}_{C}$$

(ι, ϕ) is an embedding if ι is injective (12)

and equalities replace inequalities.

Given (M, E) model of ZFC

and B s.t $(M,E) \models B$ is a ba

$$M^B = \{ f : f : M^B \to B \}^M$$

$f_\psi(\sigma_1 \ldots \sigma_m)$ is some τ s.t $(M,E) \models [\![\psi(\sigma_1 \ldots \sigma_m, \tau)]\!]^{M^B}_{RO(B)} = 1_{RO(B)}$

for $R_\phi \in \in_{\Delta_0}$ $f_\psi \in \in_{\Delta_0}$ $c \in \Delta_0$

$$[\![R_\phi(\sigma_1 \ldots \sigma_m)]\!]^{M^B}_B = [\![\phi(\sigma_1 \ldots \sigma_m)]\!]^{M^B}_{RO(B)^M}$$

Let B be a ba in M (13)

and $\dot{\kappa} \in M^{\mathsf{B}}$ be s.t.

$$(M^{\mathsf{B}}, E) \models [\![\dot{\kappa} \text{ is a regular cardinal}]\!]_{\mathsf{B}}^{M^{\mathsf{B}}} = 1_{RO(\mathsf{B})^M}$$

$$H_{\dot{\kappa}}^{\mathsf{B}} = \{\tau \in M^{\mathsf{B}} : [\![\cancel{\tau \in H_{\dot{\kappa}}}\, |\mathrm{trcl}(\{\tau\})| \leq \dot{\kappa}]\!]_{\mathsf{B}}^{\cancel{M}M^{\mathsf{B}}} = 1_{\mathsf{B}}\}$$

if M is a proper class

it is actually the case that for some $\alpha \in \mathrm{Ord}$

~~there are~~ for any $\tau \in H_{\dot{\kappa}}^{\mathsf{B}}$ ~~there~~ is $\sigma \in M_{\alpha}^{\mathsf{B}}$

s.t. $[\![\tau = \sigma]\!] = 1_{\mathsf{B}}$. Wlog ~~if~~ we can assume $H_{\dot{\kappa}}^{\mathsf{B}}$ is a set always.

$$\left(H^{B}_{\dot{K})}, \in^{M^B}_{\Delta_0}\right) \prec_1 \left(M^{B}, \in^{M^B}_{\Delta_0}\right)$$

if G is V-generic for B

$$H^{B}_{\dot{K}}[G] = \left(\cancel{H}_{\dot{K}_G}^{V[G]}, \in^{V[G]}_{\Delta_0}\right) \prec_2 \left(V[G], \in^{V[G]}_{\Delta_0}\right)$$

Cor if $G \in St(B)$ $\left(H^{B}_{\dot{K}}/_{G}, \in^{B}_{\Delta_0}/_{G}\right) \prec_1 \left(V^{B}/_{G}, \in^{B}_{\Delta_0}/_{G}\right)$

Given $(M, E) \models ZFC$ (15)

the generic multiverse over M is given by

$$\left\{ \left(H_{\check\kappa}^{M^B}/G \,,\ \in_{\Delta_0}^{M^B}/G \right) : \begin{array}{l} B \in M \quad M \models B \text{ is a ba} \\ G \in St(B) \quad [\![\check\kappa \text{ is a cardinal}]\!] = 1_{RO(B)^M} \end{array} \right\}$$

$$\subseteq V$$

Forcing theorem

Let B, M ... For $b \in B$

TFAE

- $b \leq [\![\varphi(\tau_1 \ldots \tau_n)]\!]$
- $\{G \in St(B) : M^B/G \models \varphi([\tau_1]_G \ldots [\tau_n]_G)\}$ is dense in N_b
- $\forall G$ uf. $b \in G$ $M^B/G \models \varphi([\tau_1]_G \ldots [\tau_n]_G)$

The forcing theorem holds as well if $\mathcal{M}^B$ is replaced by $H_\kappa^{\mathcal{M}^B}$ (17)

More generally the following is the case

<u>Def</u>: A B-valued model $\mathcal{M}$ is full if for any $\varphi(x_0 \ldots x_n)$ and $\tau_1 \ldots \tau_n \in \mathcal{M}$

$$[\![\exists x_0 \, \varphi(x_0, \tau_1 \ldots \tau_n]\!]_B^{\mathcal{M}} = [\![\varphi(\sigma, \tau_1 \ldots \tau_n)]\!]_B^{\mathcal{M}} \text{ for}$$

some $\sigma \in \mathcal{M}$.

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<u>Thm</u>: If a B-valued model M is full then for any $\varphi(x_1 \dots x_n)$ and $\tau_1 \dots \tau_n \in M$ ~~the~~ the following holds:

TFAE

(i) $M/G \models \varphi([\tau_1]_G \dots [\tau_n]_G)$

(ii) $[\![\varphi(\tau_1 \dots \tau_n)]\!] \in G$

TFAE for $b \in B$

(i) $[\![\varphi(\tau_1 \dots \tau_n)]\!] \geq b$

(ii) $\{G \in N_b : M/G \models \varphi([\tau_1]_G \dots [\tau_n]_G)\}$ is dense in $N_b = \{G : b \in G\}$

Def: Given $\mathcal{M}$ a B-valued model and $F \subseteq B$ a filter

$\mathcal{M}/_F$ is a $B/_F$-valued model

with elements $[\tau]_F = \{\sigma \in M : [\![\sigma = \tau]\!] \in F\}$

$$R([\tau_1]_F \cdots [\tau_n]_F) = \left[[\![R(\tau_1 \cdots \tau_n)]\!] \right]_F \in B/_F$$

$$f([\tau_1]_F \cdots [\tau_n]_F) = [f(\tau_1 \cdots \tau_n)]_F$$

Thm if $(M, E) \models ZFC$

$\left(M^B, \in_{\Delta_0}^{M^B}\right)$ is a full $RO(B)^M$-model for $\in_{\Delta_0}$

$\left(H_\kappa^{M^B}, \in_{\Delta_0}^{M^B}\right)$ is also full for $\in_{\Delta_0}$

$[\![\tau = \sigma]\!] = 1_B$

if $F = \{1_B\}$

$[\tau]_F = [\sigma]_F$

~~Given $\mathcal{M}$~~

and $(\iota, \phi) : \mathcal{M} \to \mathcal{N}$ is a boolean morphism

$\iota/_F : \mathcal{M}/_F \to \mathcal{N}/_F$