map. If U is open and closed in Y then $f^{-1}(U)$ is open and closed in X which means that $f^{-1}(U) = \emptyset$ or X and $U = \emptyset$ or Y. Thus Y is connected.

9.5 Corollary

If X and Y are homeomorphic topological spaces then X is connected if and only if Y is connected.

From Theorem 9.4 we deduce that the circle S^1 is connected since there is a continuous surjective map $f\colon [0,1] \to S^1$ given by $f(t) = (\cos(2\pi t), \sin(2\pi t)) \in S^1 \subseteq \mathbb{R}^2$.

To prove that intervals in $\mathbb{R}$ of the form $[a,b)$, $(a,b]$ and (a,b) are connected we make use of the next result.

9.6 Theorem

Suppose that $\{ Y_j ; j \in J \}$ is a collection of connected subsets of a space X. If $\bigcap_{j\in J} Y_j \neq \emptyset$ then $Y = \bigcup_{j\in J} Y_j$ is connected.

Proof Suppose that U is a non-empty open and closed subset of Y. Then $U \cap Y_i \neq \emptyset$ for some $i \in J$ and $U \cap Y_i$ is both open and closed in Y_i. But Y_i is connected so $U \cap Y_i = Y_i$ and hence $Y_i \subseteq U$. The set Y_i intersects every other Y_j, $j \in J$ and so U also intersects every Y_j, $j \in J$. By repeating the argument we deduce that $Y_j \subseteq U$ for all $j \in J$ and hence $U = Y$.

That the subsets $[a,b)$, $(a,b]$ and (a,b) of $\mathbb{R}$ are connected follows from Theorem 9.3, Corollary 9.5 and the fact that

$$[a,b) = \bigcup_{n\geq 1} [a, b - (b-a)/2^n]$$

etc. Similarly it follows that $\mathbb{R}$ itself and intervals of the form $[a,\infty)$, $(-\infty,b]$, $(-\infty,b)$, (a,∞) are connected.

The final result that we shall prove concerns products of connected spaces.

9.7 Theorem

Let X and Y be topological spaces. Then X and Y are connected if and only if $X \times Y$ is connected.

Proof Suppose that X and Y are connected. Since $X \cong X \times \{ y \}$ and $Y \cong \{ x \} \times Y$ for all $x \in X$, $y \in Y$ we see that $X \times \{ y \}$ and $\{ x \} \times Y$ are connected. Now $(X \times \{ y \}) \cap (\{ x \} \times Y) = \{ (x,y) \} \neq \emptyset$ and so $(X \times \{ y \}) \cup (\{ x \} \times Y)$ is connected by Theorem 9.6. We may write $X \times Y$ as

$$X \times Y = \bigcup_{x \in X} ((X \times \{y\}) \cup (\{x\} \times Y))$$

for some fixed $y \in Y$. Since $\bigcap_{x \in X} ((X \times \{y\}) \cup (\{x\} \times Y)) \neq \emptyset$ we deduce that $X \times Y$ is connected.

Conversely, suppose that $X \times Y$ is connected. That X and Y are connected follows from Theorem 9.4 and the fact that $\pi_X: X \times Y \to X$ and $\pi_Y: X \times Y \to Y$ are continuous surjective maps.

From the above results we see that R^n is connected. In the exercises we shall see that S^n is connected for $n \geq 1$ and also that RP^n is connected.

9.8 Exercises

(a) Prove that the set of rational numbers $Q \subseteq R$ is not a connected set. What are the connected subsets of Q?

(b) Prove that a subset of R is connected if and only if it is an interval or a single point. (A subset of R is called an *interval* if A contains at least two distinct points, and if $a, b \in A$ with $a < b$ and $a < x < b$ then $x \in A$.)

(c) Let X be a set with at least two elements. Prove
(i) If X is given the discrete topology then the only connected subsets of X are single point subsets.
(ii) If X is given the concrete topology then every subset of X is connected.

(d) Which of the following subsets of R^2 are connected?
$\{x; \|x\| < 1\}$, $\{x; \|x\| > 1\}$, $\{x; \|x\| \neq 1\}$.
Which of the following subsets of R^3 are connected?
$\{x; x_1^2 + x_2^2 - x_3^2 = 1\}$, $\{x; x_1^2 + x_2^2 + x_3^2 = -1\}$,
$\{x; x_1 \neq 1\}$.

(e) Prove that a topological space X is connected if and only if each continuous mapping of X into a discrete space (with at least two points) is a constant mapping.

(f) A is a connected subspace of X and $A \subseteq Y \subseteq \bar{A}$. Prove that Y is connected.

(g) Suppose that Y_0 and $\{Y_j; j \in J\}$ are connected subsets of a space X. Prove that if $Y_0 \cap Y_j \neq \emptyset$ for all $j \in J$ then $Y = Y_0 \cup (\bigcup_{j \in J} Y_j)$ is connected.

(h) Prove that $R^{n+1} - \{0\}$ is connected if $n \geq 1$. Deduce that S^n and