# Regular coordinate systems for Schwarzschild and other spherical spacetimes

Karl Martel and Eric Poisson

# Regular coordinate systems for Schwarzschild and other spherical spacetimes

Karl Martel and Eric Poisson
*Department of Physics, University of Guelph, Guelph, Ontario N1G 2W1, Canada*

The continuation of the Schwarzschild metric across the event horizon is a well-understood problem discussed in most textbooks on general relativity. Among the most popular coordinate systems that are regular at the horizon are the Kruskal–Szekeres and Eddington–Finkelstein coordinates. Our first objective in this paper is to popularize another set of coordinates, the Painlevé–Gullstrand coordinates. These were first introduced in the 1920s, and have been periodically rediscovered since; they are especially attractive and pedagogically powerful. Our second objective is to provide generalizations of these coordinates, first within the specific context of Schwarzschild spacetime, and then in the context of more general spherical spacetimes. © *2001 American Association of Physics Teachers.*

## I. INTRODUCTION

The difficulties of the Schwarzschild coordinates $(t,r,\theta,\phi)$ at the event horizon of a nonrotating black hole provide a vivid illustration of the fact that in general relativity, the meaning of the coordinates is not independent of the metric tensor $g_{\alpha\beta}$. The Schwarzschild spacetime, whose metric is given by (we use geometrized units, so that $c = G = 1$)

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2\,d\Omega^2,$$
$$f = 1 - 2M/r, \tag{1.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$, indeed gives one of the simplest examples of the failure of coordinates which have an obvious interpretation in one region of the spacetime (the region for which $r \gg 2M$), but not in another (the region for which $r \leqslant 2M$). Understanding this failure of the ''standard'' coordinate system is one of the most interesting challenges in the study of general relativity. Overcoming this obstacle is one of the most rewarding experiences associated with learning the theory.

Most textbooks on general relativity[1–3] discuss the continuation of the Schwarzschild solution across the event horizon, either via the Kruskal–Szekeres (KS) coordinates, or via the Eddington–Finkelstein (EF) coordinates; both coordinate systems produce a metric that is manifestly regular at $r = 2M$. The main purpose of this paper is to show that useful alternatives exist. One of them, the Painlevé–Gullstrand (PG) coordinates, are especially simple and attractive, and we will consider them in detail. We will also generalize them into a one-parameter family of coordinate systems, and show that the EF and PG coordinates are members of this family.

In a pedagogical context, the KS coordinates come with several drawbacks. First, the explicit construction of the KS coordinates is relatively complicated, and must be carried out in a fairly long series of steps. Second, the fact that $r$ is only implicitly defined in terms of the KS coordinates makes working with them rather difficult. Third, the manifold covered by the KS coordinates, with its two copies of each surface $r =$ constant, is unnecessarily large for most practical applications; while the extension across the event horizon is desirable, the presence of another asymptotic region (for

which $r \gg 2M$) often is not. While the KS coordinates are not to be dismissed out of hand—they do play an irreplaceable role in black-hole physics, and they should never be left out of a solid education in general relativity—we would advocate, for pedagogical purposes and as a first approach to this topic, the construction of simpler coordinate systems for extending the Schwarzschild spacetime across the event horizon.

A useful alternative are the EF coordinates $(v,r,\theta,\phi)$, in which the metric takes the form

$$ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2. \tag{1.2}$$

The new time coordinate $v$ is constant on ingoing, radial, null geodesics ($r$ decreases, $\theta$ and $\phi$ are constant); it is related to the Schwarzschild time $t$ by $v = t + r^*$, where

$$r^* = \int \frac{dr}{f} = r + 2M \ln\left|\frac{r}{2M} - 1\right|. \tag{1.3}$$

The metric of Eq. (1.2) is regular across the event horizon. While its nondiagonal structure makes it slightly harder to work with than the metric of Eq. (1.1), the fact that $r$ appears explicitly as one of the coordinates makes it much more convenient than the KS version of the Schwarzschild metric. We believe that in a pedagogical context, the Eddington–Finkelstein coordinates should be introduced before the KS coordinates.

Our first objective in this article is to popularize another set of coordinates for Schwarzschild spacetime, and propose this system as a useful alternative to the EF coordinates. These are the Painlevé–Gullstrand (PG[4,5]) coordinates $(T,r,\theta,\phi)$. They are constructed and discussed in Sec. II. Our second objective is to provide generalizations of this coordinate system. In Sec. III we discuss a one-parameter family of PG-like coordinates for Schwarzschild spacetime. To the best of our knowledge this family was first discovered by Kayll Lake in 1994,[6] but a related family of coordinates was previously discussed by Gautreau and Hoffmann.[7–9] We show in Sec. III that the PG and EF coordinates are both members of Lake's family. In Sec. IV we generalize this family of coordinate systems to other spherical (and static) spacetimes; equivalent coordinates were constructed, in a two-dimensional context, by Corley and Jacobson.[10] In Sec.
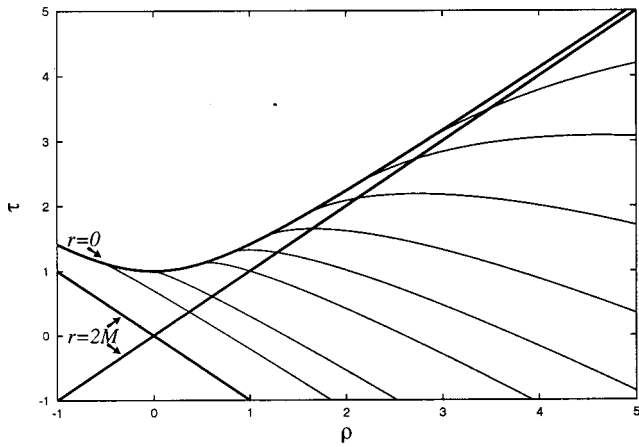
Fig. 1. Surfaces $T=$ constant in a Kruskal diagram. The vertical and horizontal axes correspond to the Kruskal $\tau$ and $\rho$ coordinates, respectively; these are defined in the Appendix. The thick, diagonal lines represent the two copies of the surface $r=2M$; the future horizon is oriented at $+45$ degrees, while the past horizon is oriented at $-45$ degrees. The thick, hyperbolic line represents the curvature singularity at $r=0$. The thin lines represent the surfaces $T=$ constant. From the bottom up we display the surfaces $T=-2M$, $T=0$, $T=2M$, $T=3M$, $T=4M$, $T=5M$, $T=6M$, and $T=7M$.

V we look back at our coordinates, and offer some additional comments regarding their construction. In the Appendix we relate these coordinate systems to the KS coordinates, and provide details regarding the spacetime diagrams of Figs. 1 and 2.

## II. PAINLEVÉ–GULLSTRAND COORDINATES

It is often a good strategy, when looking for regular coordinate systems, to anchor the coordinates to a specific family of freely moving observers.[11] We shall employ this strategy
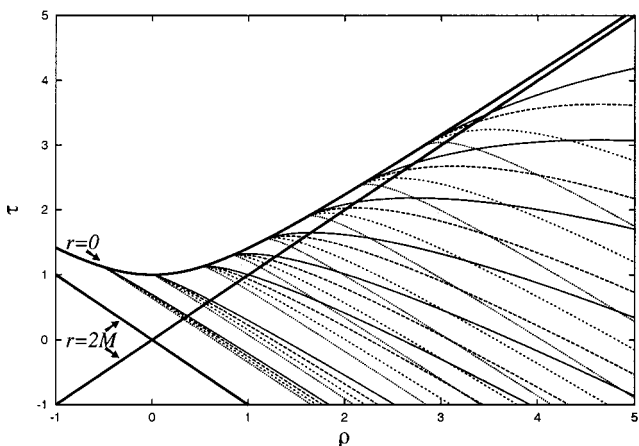


Fig. 2. Surfaces $T=$ constant in a Kruskal diagram. The axes and the meaning of the thick lines are explained in the caption of Fig. 1. The eight bundles of thin lines refer to the same values of $T$ as in Fig. 1, from $T=-2M$ (bottom bundle) to $T=7M$ (top bundle). Within a single bundle, each of the four lines come with a distinct value of the parameter $p$. The solid line corresponds to $p=1$, the long-dashed line corresponds to $p=\frac{3}{4}$, the short-dashed line corresponds to $p=\frac{1}{2}$, and the dotted line corresponds to $p=\frac{1}{4}$.

throughout this paper. The following derivation of the PG coordinates can be found in the book by Robertson and Noonan.[12] Other derivations can be found in Refs. 7, 8, and 13, in which the PG coordinates were independently rediscovered.

We consider observers which move along ingoing, radial, timelike geodesics of the Schwarzschild spacetime ($r$ decreases, $\theta$ and $\phi$ are constant). It is easy to check that in the standard coordinates of Eq. (1.1), the geodesic equations can be expressed in first-order form as

$$\dot{t}=\frac{\widetilde{E}}{f}, \quad \dot{r}^2+f=\widetilde{E}^2, \tag{2.1}$$

where an overdot denotes differentiation with respect to the observer's proper time, and $\widetilde{E}=E/m$ is the observer's (conserved) energy per unit rest mass. (For a derivation, see Chap. 11 of Ref. 1, Chap. 25 of Ref. 2, or Chap. 6 of Ref. 3.) We assume that $\dot{r}<0$, and the energy parameter is related to the observer's initial velocity $v_\infty$—the velocity at $r=\infty$—by

$$\widetilde{E}=\frac{1}{\sqrt{1-v_\infty^2}}. \tag{2.2}$$

In this section we specialize to the particular family of observers characterized by $\widetilde{E}=1$; our observers are all starting at infinity with a zero initial velocity: $v_\infty=0$. For these observers, the geodesic equations reduce to $\dot{t}=1/f$ and $\dot{r}=-\sqrt{1-f}$. We notice that $u_\alpha$, the covariant components of the observer's four-velocity, whose contravariant components are $u^\alpha=(\dot{t},\dot{r},0,0)$, is given by $u_\alpha=(-1,-\sqrt{1-f}/f,0,0)$. This means that $u_\alpha$ is equal to the gradient of some time function $T$:

$$u_\alpha=-\partial_\alpha T, \tag{2.3}$$

where

$$T=t+\int\frac{\sqrt{1-f}}{f}dr. \tag{2.4}$$

Integration of the second term is elementary, and we obtain

$$T=t+4M\left(\sqrt{r/2M}+\frac{1}{2}\ln\left|\frac{\sqrt{r/2M}-1}{\sqrt{r/2M}+1}\right|\right). \tag{2.5}$$

This shall be our new time coordinate, and $(T,r,\theta,\phi)$ are nothing but the PG coordinates. It should be clear that the key to the construction of the PG coordinates is the fact that the four-velocity can be expressed as in Eq. (2.3). In Sec. V we will explain how this equation comes about.

Going back to Eq. (2.4), we see that $dt=dT-f^{-1}\sqrt{2M/r}dr$. Substituting this into Eq. (1.1) gives

$$ds^2=-fdT^2+2\sqrt{2M/r}dTdr+dr^2+r^2d\Omega^2. \tag{2.6}$$

This is the Schwarzschild metric in the PG coordinates. An equivalent way of expressing this is

$$ds^2=-dT^2+(dr+\sqrt{2M/r}dT)^2+r^2d\Omega^2. \tag{2.7}$$

This metric is manifestly regular at $r=2M$, in correspondence with the fact that our observers do not consider this surface to be in any way special. (The metric is of course still singular at $r=0$.) While the metric is now nondiagonal, it has a remarkably simple form. It is much simpler than the

Kruskal–Szekeres metric, and we believe that it provides a useful alternative to the Eddington–Finkelstein form of the metric, Eq. (1.2).

In Fig. 1 we show several surfaces $T=$ constant in a Kruskal diagram. The construction is detailed in the Appendix. The diagram makes it clear that the PG coordinates do not extend inside the past horizon of the Schwarzschild spacetime—the ''white-hole region'' is not covered. The reason for this is that our observers fall inward from infinity and end up crossing the *future*, but not the *past*, horizon. By reversing the motion (choosing the opposite sign for $\dot{r}$), we would obtain alternative coordinates that extend within the past horizon but do not cover the black-hole region of the spacetime. While the PG coordinates do not cover the entire KS manifold, they do cover the two most interesting regions of the maximally extended Schwarzschild spacetime.

Equations (2.6) and (2.7) reveal the striking property that the surfaces $T=$ constant are intrinsically flat: Setting $dT=0$ returns $ds^2=dr^2+r^2d\Omega^2$, which is the metric of flat, three-dimensional space in spherical polar coordinates. The information about the spacetime curvature is therefore entirely encoded in the ''shift vector,'' the off-diagonal component of the metric tensor. We consider this aspect of the PG coordinates to be their most interesting property.

We note that it is possible to construct PG-like coordinates for the nonspherical Kerr spacetime. This was carried out by C. Doran in a recent paper.[14]

## III. GENERALIZATION TO OTHER OBSERVERS

It is easy to generalize the preceding construction to other families of freely moving observers. In this section we consider families such that $\tilde{E}$ is the same for all observers within the family, but not equal to unity (as in Sec. II). Each family is therefore characterized by its unique value of the energy parameter. We find it convenient to use instead the parameter $p$, related to the energy and initial-velocity parameters by

$$p=\frac{1}{\tilde{E}^2}=1-v_\infty^2. \qquad (3.1)$$

We take $p$ to be in the interval $0<p\leq1$, with $p=1$ taking us back to the PG coordinates.[15] To each value of $p$ in this interval corresponds a family of observers, and a distinct coordinate system. We are therefore constructing a one-parameter family of PG-like coordinates for Schwarzschild spacetime.

With the geodesic equations now given by $\dot{t}=1/(\sqrt{p}f)$ and $\dot{r}=-\sqrt{1-pf}/\sqrt{p}$, we find that $u_\alpha$ is now equal to a constant times the gradient of a time function $T$:

$$u_\alpha=-\frac{1}{\sqrt{p}}\partial_\alpha T, \qquad (3.2)$$

with

$$T=t+\int\frac{\sqrt{1-pf}}{f}dr. \qquad (3.3)$$

Integration of the second term doesn't present any essential difficulties, and we obtain

$$T=t+2M\left(\frac{1-pf}{1-f}+\ln\left|\frac{1-\sqrt{1-pf}}{1+\sqrt{1-pf}}\right|\right.$$
$$\left.-\frac{1-p/2}{\sqrt{1-p}}\ln\left|\frac{\sqrt{1-pf}-\sqrt{1-p}}{\sqrt{1-pf}+\sqrt{1-p}}\right|\right). \qquad (3.4)$$

This shall be our new time coordinate. In Sec. V we will return to the question of the origin of Eq. (3.2).

With $dt$ now equal to $dT-f^{-1}\sqrt{1-pf}dr$, we find that the Schwarzschild metric takes the form

$$ds^2=-fdT^2+2\sqrt{1-pf}dTdr+pdr^2+r^2d\Omega^2, \qquad (3.5)$$

or

$$ds^2=-\frac{1}{p}dT^2+p\left(dr+\frac{1}{p}\sqrt{1-pf}dT\right)^2+r^2d\Omega^2. \qquad (3.6)$$

This metric is still regular at $r=2M$, although it is now slightly more complicated than the PG form.

In Fig. 2 we show several surfaces $T=$ constant in a Kruskal diagram, for several values of $p$. This construction is detailed in the Appendix.

In this generalization of the PG coordinates, the surfaces $T=$ constant are no longer intrinsically flat. The induced metric is now $ds^2=pdr^2+r^2d\Omega^2$, and although the factor of $p$ in front of $dr^2$ looks innocuous, it is sufficient to produce a curvature. It may indeed be checked that the Riemann tensor associated with this metric is nonzero. The only nonvanishing component is $R^\phi_{\theta\phi\theta}=-(1-p)/p$, and $R^{abcd}R_{abcd}=4(1-p)^2/(pr^2)^2$.

It is instructive to go back to Eq. (3.4) and check that in the limit $p\to1$, $T$ reduces to the expression of Eq. (2.5). (This must be done as a limiting procedure, because $T$ is ambiguous for $p=1$.) Taking the limit gives

$$\lim_{p\to1}T=t+2M\left(\frac{2}{\sqrt{1-f}}+\ln\left|\frac{1-\sqrt{1-f}}{1+\sqrt{1-f}}\right|\right), \qquad (3.7)$$

which is indeed equivalent to Eq. (2.5). The PG coordinates are therefore a member of our one-parameter family.

Another interesting limit is $p\to0$, which corresponds to $\tilde{E}\to\infty$, or $v_\infty\to1$. In this limit, our observers start at infinity with a velocity nearly equal to the speed of light. Starting from Eq. (3.4) we have

$$\lim_{p\to0}T=t+2M\left(\frac{1}{1-f}+\ln\left|\frac{f}{1-f}\right|\right)=t+r^*, \qquad (3.8)$$

where we have compared with Eq. (1.3). Thus, $T=v$ in the limit $p\to0$, and our generalized PG coordinates reduce to the Eddington–Finkelstein coordinates of Eq. (1.2). This is not entirely surprising, in view of the fact that our observers become light-like in this limit. The EF coordinates, therefore, are also a (limiting) member of our one-parameter family.

We have constructed an interpolating family of coordinate systems for Schwarzschild spacetime; as the parameter $p$ varies from 1 to 0, the coordinates go smoothly from the Painlevé–Gullstrand coordinates to the Eddington–Finkelstein coordinates. This one-parameter family of coordinate systems was first discovered by Kayll Lake,[6] but a related family of coordinates, corresponding to $p>1$, was previously introduced by Gautreau and Hoffmann.[7-9] Lake obtained the new coordinates by solving the Einstein field

equations for a vacuum, spherical spacetime in a coordinate system involving $r$ and an arbitrary time $T$. The intimate relation between his coordinates and our families of freely moving observers remained unnoticed by him.

## IV. GENERALIZATION TO OTHER SPACETIMES

The coordinates constructed in the previous two sections can be generalized to other static and spherically symmetric spacetimes. In the usual Schwarzschild coordinates, we write the metric as

$$ds^2 = -e^{2\psi}f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2, \tag{4.1}$$

where $f$ and $\psi$ are two arbitrary functions of $r$. Under the stated symmetries, Eq. (4.1) gives the most general form for the metric. We assume that the spacetime is asymptotically flat, so that $f \to 1$ and $\psi \to 0$ as $r \to \infty$. If the spacetime possesses a regular event horizon at $r = r_0$, then $f(r_0) = 0$ and $\psi$ must be nonsingular for all values of $r \neq 0$.

The geodesic equations are now

$$\dot{t} = \frac{\widetilde{E}}{e^{2\psi}f}, \quad \dot{r}^2 + f = e^{-2\psi}\widetilde{E}^2, \tag{4.2}$$

where $\widetilde{E}$ is still the conserved energy per unit rest mass. Reintroducing $p = 1/\widetilde{E}^2$, we find that the covariant components of the four-velocity can be again expressed as in Eq. (3.2), with a time function $T$ now given by

$$T = t + \int \frac{\sqrt{e^{-2\psi} - pf}}{f} dr. \tag{4.3}$$

The second term can be integrated if $f$ and $\psi$ are known. Rewriting the metric of Eq. (4.1) in terms of $dT$ yields

$$ds^2 = -fe^{2\psi}dT^2 + 2e^{2\psi}\sqrt{e^{-2\psi} - pf}dTdr + pe^{2\psi}dr^2$$
$$+ r^2 d\Omega^2, \tag{4.4}$$

or

$$ds^2 = -\frac{1}{p}dT^2 + pe^{2\psi}\left(dr + \frac{1}{p}\sqrt{e^{-2\psi} - pf}dT\right)^2$$
$$+ r^2 d\Omega^2. \tag{4.5}$$

This metric is manifestly regular at an eventual event horizon, at which $f$ vanishes.

The surfaces $T = $ constant have an induced metric given by $ds^2 = pe^{2\psi}dr^2 + r^2 d\Omega^2$. Unless $\psi = 0$ and $p = 1$, these surfaces are not intrinsically flat.[16]

## V. FINAL COMMENTS

In all the cases considered in Secs. II–IV, the construction of our coordinate systems relied on the key fact that the four-velocity could be expressed as $u_\alpha = -\partial_\alpha T/\sqrt{p}$, with $p$ a constant. [This is Eq. (3.2), and in Sec. II, $p$ was set equal to unity.] This property is remarkable, and it seems to follow quite accidentally from the equations of motion. There is of course no accident, but the point remains that not every four-velocity vector can be expressed in this form.

A standard theorem of differential geometry (for example, see Appendix B of Ref. 3) states that for $u_\alpha$ to admit the form of Eq. (3.2), it must satisfy the equations $u^\alpha_{;\beta}u^\beta = 0$ and $u_{[\alpha;\beta}u_{\gamma]} = 0$, in which a semicolon denotes covariant differ-

entiation and the square brackets indicate complete antisymmetrization of the indices. The second equation states that the world lines are everywhere orthogonal to a family of spacelike hypersurfaces, the surfaces of constant $T$. This ensures that the four-velocity can be expressed as $u_\alpha = -\mu \partial_\alpha T$, for some *function* $\mu(x^\alpha)$. In general, this function is not a constant, and we do not yet have Eq. (3.2). For this we need to impose also the first equation, which states that the motion is geodesic. When both equations hold, we find that $\mu = $ constant, and this gives us Eq. (3.2).

In our constructions, we have enforced the geodesic equation by selecting freely moving observers. By selecting *radial* observers, we have also enforced the condition that the geodesics be hypersurface orthogonal. Our strategy for constructing coordinate systems is therefore limited to radial, freely moving observers in static, spherically symmetric spacetimes; it may not work for more general motions and/or more general spacetimes.

## APPENDIX A: KRUSKAL DIAGRAMS

The Kruskal diagrams of Figs. 1 and 2 are constructed as follows.

From the Schwarzschild coordinates $t$ and $r$ we define two null coordinates, $u = t - r^*$ and $v = t + r^*$, where $r^*$ is given by Eq. (1.3). From these we form the null KS coordinates, $V = e^{v/4M}$ and $U = \mp e^{-u/4M}$, where the upper sign refers to the region $r > 2M$ of the Schwarzschild spacetime, while the lower sign refers to $r < 2M$. From this we derive the relations

$$UV = -e^{r/2M}\left(\frac{r}{2M} - 1\right) \tag{A1}$$

and

$$\frac{V}{U} = \mp e^{t/2M}. \tag{A2}$$

Timelike and spacelike KS coordinates are then defined by $V = \tau + \rho$ and $U = \tau - \rho$. In our spacetime diagrams, the $\tau$ axis runs vertically, while the $\rho$ axis runs horizontally. The future horizon is located at $U = 0$, and the past horizon is at $V = 0$. The curvature singularity is located at $UV = 1$.

We express the time function of Eq. (3.4) as

$$T = t + r^* + S(r), \tag{A3}$$

where $S(r)$ is the function of $r$ that results when the second term of Eq. (3.4) is shifted by $-r^*$, as given in Eq. (1.3); this function is regular at $r = 2M$. With this definition we have $v = T - S$, $u = T - S - 2r^*$, as well as

$$V = e^{T/4M}e^{-S/4M}, \tag{A4}$$

and

$$U = -e^{r/2M}(r/2M - 1)e^{-T/4M}e^{S/4M}. \tag{A5}$$

The surfaces $T$ = constant give rise to parametric equations of the form $V(r)$ and $U(r)$, which are obtained from Eqs. (A4) and (A5) by explicitly evaluating the function $S(r)$. In these equations, $r$ can be varied from zero to an arbitrarily large value without difficulty. The diagrams of Figs. 1 and 2 are then produced by switching to the coordinates $\bar{t}$ and $\bar{r}$ and plotting the parametric curves.

[1]B. F. Schutz, *A First Course in General Relativity* (Cambridge U. P., Cambridge, 1985).

[2]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[3]R. M. Wald, *General Relativity* (Univ. of Chicago, Chicago, 1984).

[4]P. Painlevé, ''La Mécanique classique et la théorie de la relativité,'' C. R. Acad. Sci. (Paris) **173**, 677–680 (1921).

[5]A. Gullstrand, ''Allegemeine Lösung des statischen Einkörper-problems in der Einsteinschen Gravitations theorie,'' Ark. Mat., Astron. Fys. **16** (8), 1–15 (1922).

[6]K. Lake, ''A class of quasi-stationary regular line elements for the Schwarzschild geometry'' (1994) (unpublished), posted on http://xxx.1an1.gov/abs/gr-qc/9407005.

[7]R. Gautreau and B. Hoffmann, ''The Schwarzschild radial coordinate as a measure of proper distance,'' Phys. Rev. D **17**, 2552–2555 (1978).

[8]R. Gautreau, ''Light cones inside the Schwarzschild radius,'' Am. J. Phys. **63**, 431–439 (1995).

[9]The Gautreau–Hoffmann coordinates are very similar to the coordinates of Sec. III. They also constitute a one-parameter family of coordinate systems, and their parameter $R_i$ is related to our $p$ by $R_i = 2Mp/(p-1)$. The difference lies with the fact that while their $R_i$ is not meant to be negative, our $p$ is restricted to the interval $0 < p \leq 1$. These are mutually exclusive statements. But the point remains that formally, the coordinates of Sec. III are identical to the Gautreau–Hoffmann coordinates for $R_i < 0$. Note that for $R_i > 0$ (the case considered by Gautreau and Hoffmann in Ref. 7), the surfaces of constant time extend only up to $r = R_i$; they do not reach infinity.

[10]S. Corley and T. Jacobson, ''Lattice Black Holes,'' Phys. Rev. D **57**, 6269–6279 (1998).

[11]This strategy was used by I. D. Novikov to construct yet another set of coordinates for Schwarzschild spacetime. The reference is I. D. Novikov, Doctoral dissertation, Shternberg Astronomical Institute, Moscow (1963). The Novikov coordinates are discussed in Sec. 31.4 of Ref. 2. Unlike the coordinates considered in this paper, the Novikov coordinates are *comoving* with respect to the observers to which they are attached. This means that these observers move with a constant value of Novikov's spatial coordinates.

[12]H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, 1968).

[13]P. Kraus and F. Wilczek, ''A simple stationary line element for the Schwarzschild geometry, and some applications,'' Mod. Phys. Lett. A **9**, 3713–3719 (1994).

[14]C. Doran, ''A new form of the Kerr solution,'' Phys. Rev. D **61**, 067503–067506 (2000).

[15]For $p > 1$, or $\tilde{E} < 1$, the motion does not extend to infinity but starts at a turning-point radius $r_{\max}$ defined by $f(r_{\max}) = \tilde{E}^2$. We exclude these cases because the resulting surfaces $T$ = constant would not extend beyond $r = r_{\max}$, and the new coordinates would not be defined everywhere. These cases, however, are discussed in Refs. 7 and 8. See the remark in Ref. 9.

[16]We exclude the case $\psi$ = constant $\neq 0$, because the factor $e^{2\psi}$ in front of $dt^2$ in Eq. (4.1) can then be absorbed into a rescaling of the time coordinate. In other words, there is no loss of generality involved in setting $\psi = 0$ if $\psi$ is initially an arbitrary constant.

---

## ALMOST ALWAYS WRONG

The work of real science is hard and often for long intervals frustrating. You have to be a bit compulsive to be a productive scientist. Keep in mind that new ideas are commonplace, and almost always wrong. Most flashes of insight lead nowhere; statistically, they have a half-life of hours or maybe days. Most experiments to follow up the surviving insights are tedious and consume large amounts of time, only to yield negative or (worse!) ambiguous results.

Edward O. Wilson, ''Scientists, Scholars, Knaves and Fools,'' American Scientist **86** (6), 6 (1998).