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Light bending and perihelion precession: A unified approach

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The standard General Relativity results for precession of particle orbits and for bending of null rays are derived as special cases of perturbation of a quantity that is conserved in Newtonian physics, the Runge–Lenz vector. First, this method is applied to give a derivation of these General Relativity effects for the case of the spherically symmetric Schwarzschild geometry. Then the lowest order correction due to an angular momentum of the central body is considered. The results obtained are well known, but the method used is rather more efficient than that found in the standard texts, and it provides a good occasion to use the Runge–Lenz vector beyond its standard applications in Newtonian physics.

I. INTRODUCTION

Light bending and perihelion precession are the two most important effects on orbits caused by the General Relativity corrections to the Newtonian gravitational field of the sun. The standard derivation treats these two effects in different ways, without any apparent connection between them. Yet, in the usual Schwarzschild coordinates they are both due to the same, single relativistic correction to the Newtonian potential, so it is of some interest to use the same method to derive both effects.

The key to the present unified treatment is the Runge–Lenz vector. In Newtonian physics, where the two effects are absent, this vector is constant and points from the center of attraction to the orbit's perihelion.[1] Its nonconstancy in General Relativity therefore is a measure of either effect. The Runge–Lenz vector was established as a useful tool by 1924 at the latest, but it did not become popular until the 1960s.[2]

Since then a number of papers that exploit its advantages have graced the pages of this Journal,[3] and the results to be reported here can in essence be found in earlier papers, but the unified viewpoint vis a vis General Relativity is perhaps new. In addition the ''magnetic'' gravitational effects due to a rotating central body are treated here with this method.

II. GENERAL RELATIVISTIC EQUATIONS OF MOTION (NO ROTATION)

The motion to be considered is that of a ''test particle'' of mass m that moves in the space–time exterior to the central body. If this body is nonrotating and spherically symmetric, the exterior space–time geometry is described by the Schwarzschild line element

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2d\Omega^2. \tag{2.1}$$

Here we have followed the habit of geometers to express all quantities in ''geometrized'' units. Thus t is a time variable with dimensions of length, related to the ordinary time $\mathscr{T}$ and the speed of light c by $t=c\mathscr{T}$; and the constant $M=G\mathscr{M}/c^2$ has dimensions of length and is related to the total gravitational mass $\mathscr{M}$ of the central body by c and Newton's gravitational constant G. The coordinates t, r, θ, and ϕ are one of many equally valid choices for labeling space–time points, but they can nevertheless be invariantly characterized.[4]

In General Relativity the law of motion of a particle interacting only through gravity with the central body is the geodesic equation in the geometry (2.1). As in Newtonian physics, the law of motion is invariant under rotations and time translation. These symmetries lead to conservation laws of energy and angular momentum. When expressed in terms of the Schwarzschild coordinates of (2.1) and proper time, the general relativistic conservation law has a form that is very similar to the corresponding Newtonian law. From that point on we can therefore calculate as if we were doing Newtonian physics.[5]

The geodesic equation is of course a purely geometric condition, and therefore independent of the test particle's mass m. Since the conserved energy and angular momentum are proportional to m, the ''specific'' quantities, (energy per unit mass)/c^2 and (angular momentum per unit mass)/c, are independent of m. We will use the symbols E and L for these specific quantities. Similarly, we will use the symbol ϵ for ''specific rest mass,'' that is, $\epsilon=1$ for particles of finite rest mass, and $\epsilon=0$ for photons. A parameter along the geodesic will be denoted by τ. For timelike geodesics ($\epsilon=1$) it is the proper time (converted, like t, to units of length by the factor c). For null geodesics ($\epsilon=0$) τ denotes an affine parameter.[6] Thus, along the geodesic we have $ds^2=-\epsilon\, d\tau^2$. As usual we can choose the conserved direction of L to be normal to the plane $\theta=\pi/2$. The conserved quantities then are[7]

$$E=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}, \tag{2.2}$$

$$L=r^2\frac{d\phi}{d\tau}, \tag{2.3}$$

$$\mathscr{E}\equiv\frac{1}{2}(E^2-\epsilon)=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2-\frac{\epsilon M}{r}+\frac{L^2}{2r^2}-\frac{ML^2}{r^3}. \tag{2.4}$$

Except for the presence of τ instead of t, Eqs. (2.3) and (2.4) are the same as the corresponding Newtonian equations of motion of a particle of total energy per unit mass $\mathscr{E}$ in a potential $V=-\epsilon M/r-ML^2/r^3$ of which the first term represents the usual Newtonian gravitational potential, and the second term is a relativistic correction. Thus for particles as well as for light, the relativistic motion in terms of the time parameter τ is the same as the Newtonian motion in Newtonian time t if the potential is modified by the single term

$-ML^2/r^3$. We note that no slow motion assumption or other approximation is involved in this correspondence. If we are only interested in the orbit equation, then the difference between t and τ does not matter, because either one will be eliminated in the same way in favor of ϕ via Eq. (2.3).

III. SECULAR CHANGE OF ORBITS

The Newtonian bound orbits in a $1/r$ attractive potential are ellipses, of fixed eccentricity and orientation. These orbital parameters are conveniently described by the specific Runge–Lenz vector,[8] which we define as

$$\mathbf{A}=\mathbf{v}\times\mathbf{L}-\epsilon M\mathbf{e_r}. \tag{3.1}$$

Here all boldface quantities are 3-vectors in Euclidean space, $\mathbf{e_r}$ denotes a unit vector in the $\mathbf{r}$ direction, and $\mathbf{L}$ is the vector (perpendicular to the orbital plane) corresponding to the specific angular momentum of Eq. (2.3). The time parameter is always τ so that, for example, the velocity is $\mathbf{v}=d\mathbf{r}/d\tau$, a dimensionless quantity. When $\mathbf{A}$ is constant and in the direction $\phi=0$, we have

$$\mathbf{A}\cdot\mathbf{r}=Ar\cos\phi=L^2-\epsilon Mr. \tag{3.2}$$

For particles of finite rest mass ($\epsilon=1$) and $A<M$ this equation describes bound orbits that are ellipses with eccentricity $e=A/M$, and semimajor axis $a=L^2/M(1-e^2)$ aligned with the $\phi=0$ direction. For light rays ($\epsilon=0$), the equation describes unbound, straight orbits with impact parameter $b=L^2/A$. Because these orbits are traversed at the speed of light we have $L/E=b$. In either case $\mathbf{A}$ points from the center of attraction to the orbit's perihelion.

We treat the relativistic modification ML^2/r^3 of the Newtonian potential as a perturbation[9] and compute the consequent changes in direction of $\mathbf{A}$. The rate of change of $\mathbf{A}$ is[10]

$$\frac{d\mathbf{A}}{d\tau}=\left(r^2\frac{\partial V}{\partial r}-\epsilon M\right)\frac{d\mathbf{e_r}}{d\tau}=\left(\frac{3ML^2}{r^2}\right)\frac{d\phi}{d\tau}\,\mathbf{e}_\phi. \tag{3.3}$$

The direction of $\mathbf{A}$ therefore changes with angular velocity

$$\boldsymbol{\omega}=\frac{\mathbf{A}\times\dot{\mathbf{A}}}{A^2}=\left(\frac{3ML^2}{A^2r^2}\right)\frac{d\phi}{d\tau}\,\mathbf{A}\times\mathbf{e}_\phi. \tag{3.4}$$

Assume that $\mathbf{A}$ initially points in the $\phi=0$ direction and changes slowly, then its total change when the particle moves from ϕ_1 to ϕ_2 is

$$\Delta\alpha=\int_{\phi_1}^{\phi_2}\omega\,d\tau=3ML^2\int_{\phi_1}^{\phi_2}\frac{\cos\phi\,d\phi}{Ar^2}. \tag{3.5}$$

The shape of the orbit is still approximately elliptical as per Eq. (3.2) but its orientation changes slowly. We calculate the lowest order changes due to the General Relativistic correction in V by substituting the unperturbed orbit (3.2) into Eq. (3.5):

$$\Delta\alpha=\frac{3M}{AL^2}\int_{\phi_1}^{\phi_2}(A\cos\phi+\epsilon M)^2\cos\phi\,d\phi. \tag{3.6}$$

A. Perihelion precession

For a particle in a bound orbit it is customary to find the angular change of the perihelion during one revolution (in ϕ) of the particle. Because $\mathbf{A}$ points to the perihelion, this angle is given by Eq. (3.6), when $\phi_2-\phi_1=2\pi$:

$$\Delta\alpha=\frac{3M}{L^2}\int_0^{2\pi}\frac{(A\cos\phi+M)^2}{A}\cos\phi\,d\phi$$
$$=\frac{6\pi M^2}{L^2}=\frac{6\pi M}{a(1-e^2)}=\frac{6\pi G\mathscr{M}}{a(1-e^2)c^2}. \tag{3.7}$$

This is the usual perihelion formula.

B. Light bending

Here the deflection is also given by $\Delta\alpha$ of Eq. (3.6), but ϕ changes from $-\pi/2$ to $\pi/2$ with respect to the perihelion (and of course $\epsilon=0$):

$$\Delta\alpha=\frac{3M}{L^2}\int_{-\pi/2}^{\pi/2}A\cos^3\phi\,d\phi$$
$$=\frac{4MA}{L^2}=\frac{4M}{b}=\frac{4G\mathscr{M}}{bc^2}. \tag{3.8}$$

This is the usual light deflection formula.

That the light deflection should follow from the same $O(1/r^3)$ correction to the Newtonian effective potential as the perihelion rotation may be somewhat surprising, because the deflection is frequently heuristically explained as an action of the Newtonian $O(1/r)$ potential on light. Indeed, an effective potential for dr/dt would contain $O(1/r)$ terms, but in the present choice of variables these are absent—illustrating once again the arbitrary nature of coordinates in a generally covariant theory.

IV. SLOWLY ROTATING CENTRAL BODIES

If the body is slowly rotating[11] in the ϕ direction with angular momentum $\mathscr{T}=c^3J/G$, the metric (2.1) is modified by the Lense–Thirring term,[12] $-(4J/r)\sin^2\theta\,d\phi\,dt$, and by a quadrupole term that is proportional to J^2 and describes the distortion of the body and its gravitational field. Depending on the stiffness of the body, both terms can have effects of comparable magnitude on planetary motion. To lowest order, the effects simply add together. Thus each effect can be treated separately, and because the quadrupole term contributes already in the Newtonian approximation as another correction to the potential V, it will not be further considered.

Like the quadrupole term, the Lense–Thirring term breaks the spherical symmetry, so on symmetry grounds only L, the generalized momentum conjugate to ϕ, is conserved. Nonetheless, if the body is stiff enough so that only the Lense–Thirring term is significant (that is, to first order in J), the "total angular momentum" $Q^2=p_\theta^2+\cot^2\theta\,L^2$, where p_θ is the generalized momentum conjugate to θ, is also conserved.[13] Thus the entire motion can be formulated in terms of conserved quantities, and one finds that the angular momentum $\mathbf{L}$ precesses around the z direction.[14,15] However, we will confine attention to the case when the motion is confined to the equatorial plane, and the general relativistic correction is completely described by the behavior of $\mathbf{A}$. $\mathbf{J}$ and $\mathbf{L}$ are then parallel to each other, and normal to the orbital plane.

A. Equations of motion and Runge–Lenz vector

A Lagrangian $\mathscr{L}$ for a particle or a light beam moving in the equatorial plane of the metric

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\frac{dr^2}{1-\frac{2M}{r}}$$

$$+r^2 d\Omega^2-\frac{4J}{r}\sin^2\theta\, d\phi\, dt \tag{4.1}$$

is given by

$$\mathscr{L}=\left(\frac{ds}{d\tau}\right)^2=-\left(1-\frac{2M}{r}\right)\dot{t}^{\,2}+\frac{\dot{r}^2}{1-\frac{2M}{r}}$$

$$+r^2\dot{\phi}^2-\frac{4J}{r}\dot{\phi}\dot{t}. \tag{4.2}$$

Here the dot indicates differentiation with respect to τ. Because $-ds^2=\epsilon\, d\tau^2$ along the trajectory, $\mathscr{L}$ is conserved,

$$\mathscr{L}=-\epsilon. \tag{4.3}$$

Since $\mathscr{L}$ is independent of t and ϕ we have further conserved quantities,

$$\begin{aligned}-E&=\frac{1}{2}\frac{\partial\mathscr{L}}{\partial\dot{t}}=-\left(1-\frac{2M}{r}\right)\dot{t}-\frac{2J}{r}\dot{\phi},\\ L&=\frac{1}{2}\frac{\partial\mathscr{L}}{\partial\dot{\phi}}=r^2\dot{\phi}-\frac{2J}{r}\dot{t}.\end{aligned} \tag{4.4}$$

Here the factor $\frac{1}{2}$ is introduced purely as a convention, so the conserved quantities agree more nearly with the conventional specific energy and angular momentum.

We solve for $\dot{t}$ and $\dot{\phi}$ to first order in J,

$$\dot{\phi}=\frac{L}{r^2}+\frac{2JE}{r^3},\quad \dot{t}=\frac{E}{1-\frac{2M}{r}}-\frac{2JL}{r^3} \tag{4.5}$$

and substitute into Eq. (4.3) to obtain a ''conservation of energy'' in an effective potential,

$$E^2-\epsilon=\dot{r}^2-\frac{2\epsilon M}{r}+\frac{L^2}{r^2}-\frac{2ML^2}{r^3}+\frac{4JLE}{r^3}. \tag{4.6}$$

The effective potential in this equation contains two non-Newtonian terms. The first was already encountered in Eq. (2.4) and causes the ''standard'' relativistic correction; the second is due to the Lense–Thirring addition to the metric.[16] By including these correction terms we can perform the rest of the calculation as if we were doing Newtonian physics.

Because in the rotating case there is a difference between kinematic angular momentum ($r^2\dot{\phi}$) and canonical angular momentum L, the Runge–Lenz vector $\mathbf{A}$ can be defined in various ways, but there is no difference in the precession rate one calculates from them. A convenient choice is

$$\begin{aligned}\mathbf{A}&=\mathbf{v}\times\left(\mathbf{L}-\frac{2\mathbf{J}E}{r}\right)-\epsilon M\mathbf{e_r}\\ &=\left(\frac{L^2}{r}-\epsilon M\right)\mathbf{e_r}-\dot{r}\left(L-\frac{2JE}{r}\right)\mathbf{e}_\phi\end{aligned} \tag{4.7}$$

because it simplifies the equation of motion for $\mathbf{A}$, and still gives elliptical orbits for any J when $\mathbf{A}$ is constant, as in Eq. (3.2):

$$\mathbf{A}\cdot\mathbf{e_r}=A\cos\phi=\frac{L^2}{r}-\epsilon M. \tag{4.8}$$

The equation of motion for $\mathbf{A}$ can be derived from Eqs. (4.5) and (4.6):

$$\dot{\mathbf{A}}=\left(\frac{3ML^2}{r^2}-\frac{8\epsilon MJE}{Lr}+\frac{2JE(\epsilon-E^2)}{L}\right)\dot{\phi}\mathbf{e}_\phi. \tag{4.9}$$

By substituting Eq. (4.9) into Eq. (3.4) and integrating, using (4.8) for $1/r$, we now find that the total change in $\mathbf{A}$ when the particle moves from ϕ_1 to ϕ_2 is

$$\begin{aligned}\Delta\alpha=\int_{\phi_1}^{\phi_2}&\left(\frac{3M}{AL^2}(A\cos\phi+\epsilon M)^2-\frac{8\epsilon MJE}{AL^3}\right.\\ &\left.\times(A\cos\phi+\epsilon M)+\frac{2JE(\epsilon-E^2)}{AL}\right)\cos\phi\, d\phi.\end{aligned} \tag{4.10}$$

B. Perihelion motion

To obtain the perihelion motion we evaluate Eq. (4.10) over one revolution ($\phi_1=0$, $\phi_2=2\pi$) with $\epsilon=1$. Since the particle velocity is nonrelativistic, we may set $E=1$ to the lowest order:[17]

$$\begin{aligned}\Delta\alpha&=\frac{6\pi M^2}{L^2}-\frac{8\pi JME}{L^3}\\ &=\frac{6\pi M}{a(1-e^2)}-\frac{8\pi J}{M^{1/2}(a(1-e^2))^{3/2}}\\ &=\frac{G}{c^2}\left[\frac{G\pi\mathscr{M}}{a(1-e^2)}-\frac{8\pi\mathscr{J}}{(GM)^{1/2}(a(1-e^2))^{3/2}}\right].\end{aligned} \tag{4.11}$$

The first term is the ''standard'' general relativistic precession already found in Sec. III, and the second term is due to the rotation of the central body.[18] For nearly circular orbits, we can interpret this second term as due to two causes: one is the rotation in ϕ of the ''locally nonrotating observer'' that makes the Lense–Thirring term of Eq. (4.2) disappear at the radius of the particle, an amount $4\pi J/aL$; the other is the ''differential rotation'' due to the $1/r^3$ falloff of the second non-Newtonian term in the effective potential of Eq. (4.6). This contribution causes precession by an amount $-12\pi J/aL$, in the same way as the first non-Newtonian term causes the ''standard'' precession.[19]

C. Light bending

For the effect on light we put $\epsilon=0$ in Eq. (4.10) and integrate from $\phi=-\pi/2$ to $\pi/2$ as in Sec. III B,

$$\Delta\alpha=\frac{3MA}{L^2}\frac{4}{3}-\frac{2JE^3}{AL}2=\frac{4M}{b}-\frac{4J}{b^2}=\frac{4G}{c^2}\left[\frac{\mathscr{M}}{b}-\frac{\mathscr{J}}{cb^2}\right]. \tag{4.12}$$

The effect of the central body's rotation (J) on both precession and bending is negative.[20] This is the same ''differential'' dragging effect that makes a gyroscope in the equatorial plane precess in the opposite direction to the central body's rotation.[14]

V. CONCLUSIONS

For nonrotating spherically symmetric central masses we have seen that the two important general relativistic corrections to the Newtonian gravitational motion, namely perihelion precession and light bending, follow from the same correction term in the effective radial potential; and that either effect can be viewed as a change in the Runge–Lenz vector associated with the orbit. Because both effects follow by simple evaluation from one formula [Eq. (3.6)], the effort is only about half that of the usual procedure; moreover it gives occasion to review and apply the Runge–Lenz vector. We have shown the extension of this calculation to equatorial orbits of a rotating body [Eq. (4.10)]; relativistic corrections to parabolic and hyperbolic orbits can similarly be evaluated by this method.

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[1]Although our treatment is not confined to the solar system, we use this term to denote the point of closest approach to the central body because it seems more familiar (and more etymologically consistent) than the more correct term, pericenter.

[2]For a history of the Laplace–Runge–Lenz vector, see H. Goldstein, ''Prehistory of the Runge–Lenz vector,'' Am. J. Phys. **43** (8), 737–738 (1975) and ''More on the prehistory of the Laplace or Runge–Lenz vector,'' **44**, 1123–1124 (1976).

[3]See C. E. Aguiar and M. F. Banoso, ''The Runge–Lenz Vector and Perturbed Rutherford Scattering,'' Am. J. Phys. **64** (8), 1042–1048 (1996), and the references cited therein.

[4]For example, $4\pi r^2$ is the area of the sphere $r=$const, $t=$const, and $\partial/\partial t$ is the time-like Killing vector that has unit length at infinity. See C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[5]As in Newtonian physics, it is customary in general relativity to derive the equations of motion in the Schwarzschild geometry by using all the conservation laws and identifying an effective potential in a radial energy conservation equation. For the effects we want to calculate we need the radial acceleration $d^2r/d\tau^2$, so it would be a little more straightforward to use the radial component of the geodesic equation and the conservation of angular momentum. But we follow the equivalent, customary route of finding the radial acceleration from the gradient of the effective potential.

[6]For the case of light, the affine parameter τ is defined only up to scale transformations. The quantities E and L are therefore similarly defined only up to such rescaling in this case. The final, physical results will contain only ratios of such quantities, and are therefore independent of rescaling.

[7]For a derivation see, for example, Bernard F. Schutz, *A First Course in General Relativity* (Cambridge U.P., Cambridge, 1985), p. 275 or the reference of footnote 4 on p. 656, or Sec. IV of the present paper.

[8]The Runge–Lenz vector has been defined with various factors of m by various authors. Our **A** is $(mc)^{-2}\times$ that of Ref. 10, and has the advantage that it gives a finite value for particles of finite rest mass m as well as for light $(m=0)$.

[9]This means $M/r \ll 1$, where r is a typical orbit radius; it follows if we assume $M^2/L^2 \ll 1$.

[10]See, for example, H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, MA, 1980), 2nd ed.

[11]We work only to first order in J; more precisely, we assume $J \lesssim ML$ and, as before, $M/r \sim M^2/L^2 \sim \epsilon \ll 1$, so that $J/r^2 \sim \epsilon^{3/2}$.

[12]J. Lense and H. Thirring, ''Über den Einfluss der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie,'' Phys. Z. **19**, 156–162 (1918); English translation in Gen. Relativ. Gravit. **16**, 711–750 (1984). Also see D. R. Brill and J. M. Cohen, ''Rotating Masses and Their Effects on Inertial Frames,'' Phys. Rev. **143**, 1011–1015 (1966).

[13]For the geometrical reason for the conservation of Q^2 see D. Bocaletti and G. Pucacco, ''Killing Equations in Classical Mechanics,'' Nuovo Cimento B **122**, 181–212 (1997).

[14]For a summary of all the relativistic effects on orbits see I. Ciufolini and J. A. Wheeler, *Gravitation and Inertia* (Princeton, U.P., Princeton, 1995). One of the aims of the Lense–Thirring paper cited in Ref. 12 was to integrate the equations of motion for orbits of general orientation.

[15]For a treatment using the Runge–Lenz vector, see L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), p. 336; S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), p. 230.

[16]With our assumptions as spelled out in Ref. 11 the Newtonian terms of the effective potential are of order ϵ, both non-Newtonian terms are of order ϵ^2, and typical terms that are neglected are $J^2E^2/r^4 \sim JEML/r^4 \sim \epsilon^3$, $J^2L^2/r^6 \sim \epsilon^4$, etc.

[17]Formally, this follows from the $M^2 \ll L^2$ assumption and requiring bound orbits.

[18]The last term in Eq. (4.11) changes sign if **L** is antiparallel to **J**. Both terms are of order ϵ.

[19]For nonequatorial orbits the first contribution is a precession about **J**, whereas the second contribution is a precession about **L**, proportional to $\mathbf{J}\cdot\mathbf{L}$.

[20]However, the relative contribution of J to light bending is less: We have $M/b \sim \epsilon$, but $J/b^2 \sim \epsilon^{3/2}$.

WORD PROBLEMS

Bennett's classmates hated word problems. Indeed, they hated math altogether, but they'd rather have a tooth filled than be forced to sit down and contemplate word problems. Bennett, on the other hand, placed word problems on a level with Florida's pecan pie. Word problems were delicious. He devoured them. He convinced the flabbergasted Mrs. Dixon to give him additional problems, beyond the assignments, and when she ran out of problems he created them himself. After school, when the other boys played basketball or loitered behind the Rexall drugstore to smoke and discuss girls, Bennett went home and up to his room to do word problems.

Alan Lightman, *Good Benito* (Pantheon Books, New York, 1994), p. 66.