G gp. finito

$\#\{$rapp. di G irr. non-equivalenti su $\mathbb{C}\}$

$\|$

$\#\{$classi coniugati di G$\}$ $\updownarrow$ 1-1

$\|$

$\#\{$caratteri irr. di G a coeff. in $\mathbb{C}\}$,

Siano $C_1, C_2, \dots, C_r$ classi coniugati di G.

$\chi_1, \dots \chi_r$ caratteri irr. di G su $\mathbb{C}$.

Tabelle di Caratteri di G:

	C_1	C_2	$\dots$	C_r
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	$\dots$	$\chi_1(g_r)$
χ_2	$\chi_2(g_1)$	$\chi_2(g_2)$	$\dots$	$\chi_2(g_r)$
$\vdots$	$\vdots$			
χ_r	$\chi_r(g_1)$	$\dots$	$\dots$	$\chi_r(g_r)$

$g_1 \in C_1$, $g_2 \in C_2$, $\dots$, $g_r \in C_r$

e.g.: $G = S_3$

	$\{id\}$	$\{(12),(13),(23)\}$	$\{(123),(132)\}$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Oss: Se $\mathrm{Char}\, F \mid |G|$

Teoria di rapp. di G su F:

Modular representations of G.

$\#\{$rapp. irr. di G coeff in $F\} = ?$
non-equival.

$\#\{F[G]$-mod. semplici non-isomorf.$\}$.

J rad. Jacobson di $F[G]$, $F[G]/J$ s.s

$\{F[G]/J$ -mod. semplici non-isomorf$\}$.

Osser: S un semplice $F[G]/J$ - mod.

$$\begin{tikzcd} F[G] \arrow[r, "\pi"] \arrow[rr, bend right] & F[G]/J \arrow[r, "F[G]/J\text{-mod}"] & \mathrm{End}\, S \end{tikzcd}$$

$\Rightarrow S$ è un $F[G]$-mod. semplice

- Sia M un $F[G]$-mod. semplice.

$$J(F[G]) = \bigcap_{\substack{\forall S \text{ semplice} \\ F[G]\text{-mod}}} \mathrm{Ann}_{F[G]} S.$$

$JM = \{0\}$

$\Longrightarrow M$ è un $F[G]/J$ - mod. sempl.

Siano U e V spaz. vett. $/ \mathbb{C}$

$U \otimes_{\mathbb{C}} V$ è spaz. vett. $/ \mathbb{C}$.

$\dim_{\mathbb{C}} U \otimes_{\mathbb{C}} V = \dim_{\mathbb{C}} U \cdot \dim_{\mathbb{C}} V$.

$\{u_1, \dots, u_n\}$ una base di U

$\{v_1, \dots, v_m\}$ - - - - - V

$\{u_i \otimes v_j \mid \forall 1 \leq i \leq n, 1 \leq j \leq m\}$ una base di $U \otimes_{\mathbb{C}} V$

$\dim_{\mathbb{C}} U \otimes_{\mathbb{C}} V = m \cdot n = \dim_{\mathbb{C}} \underbrace{M_{m,n}(\mathbb{C})}$.

$U \otimes_{\mathbb{C}} V \underset{\substack{\text{isom.} \\ \text{di } \mathbb{C}\text{-spaz.}}}{\cong} \mathrm{Hom}_{\mathbb{C}}(U, V)$.

$U^* = \mathrm{Hom}_{\mathbb{C}}(U, F)$

Siano $\left.\begin{array}{l} \rho : G \to GL(U) \\ \rho' : G \to GL(V) \end{array}\right\}$.

def: $\rho \otimes \rho' : G \longrightarrow GL(U \otimes_{\mathbb{C}} V)$

$\forall g \in G, u \in U, v \in V, \ \rho \otimes \rho'(g)(u \otimes v) \underset{\text{def}}{=} \rho(g) u \otimes \rho'(g) v$.

$$\forall g \in G, \quad \rho(g) \overset{\{u_1, \dots u_n\}}{\approx} (x_{ij})_{n \times n} \in GL_n(\mathbb{C})$$

$$\rho'(g) \overset{\{v_j \dots\}}{\approx} X' \in GL_m(\mathbb{C})$$

$$\rho \otimes \rho'(g) \overset{\{u_i \otimes v_j \mid \forall i, \forall j\}}{=\!=} \begin{pmatrix} x_{11}X' & x_{12}X' & \cdots & x_{1n}X' \\ x_{21}X' & & & \vdots \\ \vdots & & \ddots & \\ x_{n1}X' & \cdots & \cdots & x_{nn}X' \end{pmatrix}$$

$$\chi_{\rho \otimes \rho'}(g) = tr(\rho \otimes \rho'(g))$$
$$= \sum_{i=1}^{n} x_{ii} \cdot tr X'$$
$$= tr(x_{ij}) \cdot tr X'.$$
$$= \chi_\rho(g) \cdot \chi_{\rho'}(g)$$

$$\chi_{\rho \otimes \rho'} = \chi_\rho \cdot \chi_{\rho'}.$$

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e.g.: $\rho : G \to GL(V)$. $V/\mathbb{C}$

def. $\rho^* : G \to GL(V^*)$ } rapp. di G

$\forall g \in G$, $[\rho^*(g) f](u) := f(\rho(g^{-1})u)$

$\forall f \in V^*$

$\forall u \in V$

$\{u_1, \dots, u_n\}$ base di U

$\{f_1, \dots, f_n\}$ base duale in U^*

$$f_i(u_j) = \delta_{ij}, \quad 1 \le i, j \le n.$$

$\forall g \in G$, sia $\rho(g) \overset{\{u_1 \dots u_n\}}{=\!=} X \in Gl_n(\mathbb{C})$

$$\Rightarrow \rho^*(g) \overset{\{f_1 \dots f_n\}}{=\!=} (X^t)^{-1}$$

$$\Rightarrow \chi_\rho = \overline{\chi_{\rho^*}}$$

Siano U e V due $\mathbb{C}[G]$-mod.

In corrisp. $U \otimes_{\mathbb{C}} V$ è un $\mathbb{C}[G]$-modulo

" $\underline{U^* \otimes_{\mathbb{C}} V}$ - - - - - - - -

i.e. $\forall g \in G$, $f \in U^*$, $v \in V$

$$g \cdot (f \otimes v) = g_* f \otimes g \cdot v.$$

dove $g_* f(u) = f(g^{-1} u)$, $\forall u \in U$.

$\mathrm{Hom}_{\mathbb{C}}(U, V)$ è un $\mathbb{C}[G]$-modulo dove l'operaz. di "coeff" è definita da "$\bullet$":

$$[g\bullet\tau](u) \overset{def}{=} g\tau(g^{-1}u) \in V$$

$\forall g \in G$, ossia:

$\forall \tau \in \mathrm{Hom}_{\mathbb{C}}(U, V)$,

$\forall u \in U$,

$$\begin{tikzcd} U \arrow[r, "\tau"] \arrow[d, "g"'] & V \arrow[d, "g"] \\ U \arrow[r, "g\bullet\tau"] & V \end{tikzcd}$$

si commuta.

Si estende "$\bullet$" ad su tutti elem. di $\mathbb{C}[G]$.

Osser: $g\bullet\tau = \tau, \forall g \in G \iff \forall u \in U, \forall g \in G.$ $\tau(u) = g\cdot\tau(g^{-1}u)$

$\iff \forall g \in G, \forall u \in U,$ $g^{-1}\tau(u) = \tau(g^{-1}u)$

i.e. $\forall g \in G, u \in U,$ $g\tau(u) = \tau(gu)$

$\iff \forall x \in \mathbb{C}[G],\ x\tau(u) = \tau(xu).$

$\iff \tau \in \mathrm{Hom}_{\mathbb{C}[G]}(U, V)$

per $\tau \in \mathrm{Hom}_{\mathbb{C}}(U,V)$

<u>Lemma</u>: $\forall g \in G,\ g\cdot\tau = \tau \iff \tau \in \mathrm{Hom}_{\mathbb{C}[G]}(U,V)$.

Sia U un $\mathbb{C}[G]$-mod.

$$U^G = \{u \in U \mid gu = u,\ \forall g \in G\}$$

<u>Prop</u>: Sia $e = \frac{1}{|G|}\sum_{\forall g \in G} g \in \mathbb{C}[G]$,

allora $U^G = eU$, di conseguenza

$$\dim_{\mathbb{C}} U^G = \frac{1}{|G|}\sum_{\forall g \in G} \chi_U(g).$$

<u>Dim</u>: $\forall h \in G$,

$$he = \frac{1}{|G|}\sum_{\forall g \in G} hg = \frac{1}{|G|}\sum_{\forall g' \in G} g'$$
$$= e$$

$$e^2 = \frac{1}{|G|}\sum_{\forall g \in G} ge = \frac{|G|}{|G|}e = e.$$

$$e \in \mathrm{End}_{\mathbb{C}} U \Rightarrow U = eU \oplus (1-e)U.$$

Sia $\{\underbrace{u_1, \dots u_r}_{\text{base di } U}, \underbrace{u_{r+1}, \dots, u_n}_{\text{base di } (1-e)U}\}$ una base di U

$\forall x \in eU$, $\Rightarrow x = eu$, $u \in U \Rightarrow ex = eeu$
$= e^2 u = eu$
$= x$

$$e\big|_{eU} = id_{eU}$$

$$e\big|_{(1-e)U} = 0_{(1-e)U}$$

$$e \xrightarrow{\{u_i \mid i=1..n\}} \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & 0 \end{pmatrix} \begin{array}{l} \}\, r = \dim_{\mathbb{C}} eU \\ \\ \}\, n-r. \end{array}$$

$\operatorname{tr} e = \dim_{\mathbb{C}} eU.$

$U^G = eU$ perché:

"$\subseteq$": $u \in U^G$, $u = \dfrac{\overbrace{u + \cdots + u}^{|G|}}{|G|}$, $\forall g \in G$, $gu = u$.

$$= \frac{\sum_{\forall g \in G} gu}{|G|}$$

$$= eu \in eU.$$

"$\supseteq$": $u \in eU$, $\Rightarrow u = eu'$, $u' \in U$

$g \in G$, $gu = geu' = eu' = u$, $\Rightarrow u \in U^G$.

$\dim_{\mathbb{C}} U^G = \dim_{\mathbb{C}} eU = \operatorname{tr} e = \operatorname{tr}\left(\frac{1}{|G|} \sum g\right)$

$$= \frac{1}{|G|} \sum_{\forall g} \operatorname{tr} g = \frac{1}{|G|} \sum_{\forall g \in G} \chi_U(g) .$$ $\boxed{\text{CG}}$

Prop: Siano U e V due $\mathbb{C}[G]$-moduli. allora

$$\operatorname{Hom}_{\mathbb{C}[G]}(U,V) = \operatorname{Hom}_{\mathbb{C}}(U,V)^G$$

Dim: $\tau \in \operatorname{Hom}_{\mathbb{C}}(U,V)^G \Longleftrightarrow$

$$\forall g \in G,\ g \cdot \tau = \tau \overset{\text{Lemma}}{\Longleftrightarrow} \tau \in \operatorname{Hom}_{\mathbb{C}[G]}(U,V).$$

$\boxed{\text{CVD}}$