DEFINITION 1.16. Suppose that $\eta_1 = f(z,\bar{z})dz \wedge d\bar{z}$ is a C^∞ 2-form in the coordinate z, defined on an open set V_1. Also suppose that $\eta_2 = g(w,\overline{w})dw \wedge d\overline{w}$ is a C^∞ 2-form in the coordinate w, defined on an open set V_2. Let $z = T(w)$ define a holomorphic mapping from the open set V_2 to V_1. We say that η_1 *transforms to* η_2 *under* T if $g(w,\overline{w}) = f(T(w),\overline{T(w)})\|T'(w)\|^2$.

The above definition comes exactly from making the change of coordinates both in the function parts and the dz and $d\bar{z}$ parts of the expression, and then using the rules given above for simplifying and cancelling, noting that $\|T'(w)\|^2 = T'(w)\overline{T'(w)}$.

Again the same method is used to transport these ideas to a Riemann surface:

DEFINITION 1.17. Let X be a Riemann surface. A C^∞ *2-form* on X is a collection of C^∞ 2-forms $\{\eta_\phi\}$, one for each chart $\phi : U \to V$ in the variable of the target V, such that if two charts $\phi_i : U_i \to V_i$ (for $i = 1,2$) have overlapping domains, then the associated C^∞ 2-form η_{ϕ_1} transforms to η_{ϕ_2} under the change of coordinate mapping $T = \phi_1 \circ \phi_2^{-1}$.

Finally the same atlas remark holds again:

LEMMA 1.18. *Let X be a Riemann surface and $\mathcal{A}$ a complex atlas on X. Suppose that C^∞ 2-forms are given for each chart of $\mathcal{A}$, which transform to each other on their common domains. Then there exists a unique C^∞ 2-form on X extending these C^∞ 2-forms on each of the charts of $\mathcal{A}$.*

Problems IV.1

A. Let X be the Riemann Sphere $\mathbb{C}_\infty$, with local coordinate z in one chart and $w = 1/z$ in the other chart. Let ω be a meromorphic 1-form on X. Show that if $\omega = f(z)dz$ in the coordinate z, then f must be a rational function of z. Show further that there are no nonzero holomorphic 1-forms on $\mathbb{C}_\infty$. Where are the zeroes and poles, and the orders, of the meromorphic 1-form defined by dz? Of the 1-form dz/z?

B. Let L be a lattice in $\mathbb{C}$, and let $\pi : \mathbb{C} \to X = \mathbb{C}/L$ be the natural quotient map. Show that the local formula dz in every chart of $\mathbb{C}/L$ is a well defined holomorphic 1-form on $\mathbb{C}/L$. Show that this 1-form has no zeroes. Show that the local formula $d\bar{z}$ in every chart of $\mathbb{C}/L$ is a well defined C^∞ 1-form on $\mathbb{C}/L$.

C. Let X be a smooth affine plane curve defined by $f(u,v) = 0$. Show that du and dv define holomorphic 1-forms on X, as do $p(u,v)du$ and $p(u,v)dv$ for any polynomial $p(u,v)$. Show that if $r(u,v)$ is any rational function, then $r(u,v)du$ and $r(u,v)dv$ are meromorphic 1-forms on X. Show that $(\partial f/\partial u)du = -(\partial f/\partial v)dv$ as holomorphic 1-forms on X.

D. Let X be a smooth projective plane curve defined by a homogeneous polynomial $F(x,y,z) = 0$. Let $f(u,v) = F(u,v,1)$ define the associated smooth affine plane curve. Show that du and dv define meromorphic 1-forms on all of X, as do $r(u,v)du$ and $r(u,v)dv$ for any rational function r. Show that $(\partial f/\partial u)du = -(\partial f/\partial v)dv$ as meromorphic 1-forms on X.

E. With the notation of the previous problem, suppose that $F(x,y,z)$ has degree $d \geq 3$. Show that if $p(u,v)$ is any polynomial of degree at most $d-3$, then

$$p(u,v)\frac{du}{\partial f/\partial v}$$

defines a holomorphic 1-form on the compact Riemann surface X.

F. Suppose that X is a projective plane curve of degree d with nodes, defined by the affine equation $f(u,v)=0$. Show that if $p(u,v)$ is any polynomial of degree at most $d-3$, which vanishes at the nodes of X, then

$$p(u,v)\frac{du}{\partial f/\partial v}$$

defines a holomorphic 1-form on the resolution $\tilde{X}$ of the nodes.

G. Let X be a compact hyperelliptic Riemann surface defined by $y^2 = h(x)$, where h has degree $2g+1$ or $2g+2$ (so that X has genus g). Show that dx/y is a holomorphic 1-form on X if $g \geq 1$. Show that $p(x)dx/y$ is a holomorphic 1-form on X if $p(x)$ is a polynomial in x of degree at most $g-1$.

H. Let X be a cyclic cover of the line defined by $y^d = h(x)$. Show that $r(x,y)dx$ defines a meromorphic 1-form on X. Give criteria for when $r(x,y)dx$ is a holomorphic 1-form.

I. Let L be a lattice in $\mathbb{C}$, and let $\pi : \mathbb{C} \to X = \mathbb{C}/L$ be the natural quotient map. Show that $dz \wedge d\bar{z}$ is a well defined C^∞ 2-form on $\mathbb{C}/L$.

J. Prove Lemma 1.8.

2. Operations on Differential Forms

There are several operations which one can perform with forms to produce other forms. We briefly describe them here, and we will leave the details of most of the constructions to the reader.

Multiplication of 1-Forms by Functions. Suppose that h is a C^∞ function on a Riemann surface X, and ω is a C^∞ 1-form on X. We may define a C^∞ 1-form $h\omega$ locally, by writing $\omega = f dz + g d\bar{z}$ and declaring $h\omega$ to be $hf dz + hg d\bar{z}$. It is an immediate check that this gi[...] properties listed [...]

The Poincaré and Dolbeault Lemmas. The Poincaré and Dolbeault Lemmas address the question: when is a function equal to the derivative of another function, at least locally? More precisely, when is a 1-form ω equal to df or $\overline{\partial} f$, locally? Clearly since $ddf = 0$, a necessary condition for $\omega = df$ is that $d\omega = 0$; since $\overline{\partial} f$ has type $(0,1)$, a necessary condition for $\omega = \overline{\partial} f$ is that ω be of type $(0,1)$.

It turns out that these conditions are sufficient as well. We will not use these results in an important way, and so will not give proofs; they can be found in many texts.

PROPOSITION 2.7 (POINCARÉ'S LEMMA). *Let ω be a C^∞ 1-form on a Riemann surface X. Suppose that $d\omega = 0$ identically in a neighborhood of a point p in X. Then on some neighborhood U of p there is a C^∞ function f defined on U with $\omega = df$ on U.*

A proof can be found in [**Munkres91**]; the idea is to use path integration (which we will discuss in the next section) and show that the function $f(z) = \int_p^z \omega$ is well defined (using $d\omega = 0$) and satisfies $df = \omega$ (by the fundamental theorem of calculus).

Dolbeault's Lemma is not as elementary.

PROPOSITION 2.8 (DOLBEAULT'S LEMMA). *Let ω be a C^∞ $(0,1)$-form on a Riemann surface X. Then on some neighborhood U of p there is a C^∞ function f defined on U with $\omega = \overline{\partial} f$ on U.*

In the real analytic category a proof is elementary, and goes as follows. Write $\omega = g(z,\bar{z})d\bar{z}$. We seek a function f such that $\partial f/\partial \bar{z} = g$. If g is real analytic, then it can be expanded in a series and we may write $g = \sum_{i,j} c_{ij} z^i \bar{z}^j$. Then we may integrate term-by-term, and set $f = \sum_{i,j} c_{ij} z^i \bar{z}^{j+1}/(j+1)$.

See for example [**Forster81**] for a general proof.

Problems IV.2

A. Check that if ω is a C^∞ 1-form and h is a C^∞ function, then $h\omega$ defined as in the text is a C^∞ 1-form.

B. Prove Lemma 2.1.

C. Prove Lemma 2.2, i.e., that the wedge product of two 1-forms is a well defined 2-form.

D. Prove Lemma 2.3.

E. Prove Lemma 2.4.

F. Prove Lemma 2.5, i.e., that the pullback of a 1-form is well defined.

G. Prove that the pullback of a 2-form is well defined.

H. Let a holomorphic map $F : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by the formula $w = z^N$ for some integer $N \geq 2$, where we use z as an affine coordinate in the domain and w as an affine coordinate in the range. Compute the pullback

$F^*((1/w)dw)$ of the form $(1/w)dw$. Compute the orders of $F^*((1/w)dw)$ at all of its zeroes and poles.

I. Let X be a hyperelliptic curve defined by $y^2 = h(x)$. Let $\pi : X \to \mathbb{P}^1$ be the double covering map sending (x, y) to x. Let $\omega = \pi^*(dx/h(x))$. Compute the orders of ω at all of its zeroes and poles.

3. Integration on a Riemann Surface

We are now in a position to describe contour integration for a Riemann surface.

Paths. The concept of a 1-form is specifically designed to provide an integrand for a "contour integral" on a Riemann surface. The other ingredient of such an integral is the contour itself. This we now develop briefly; these ideas should be quite well known.

DEFINITION 3.1. A *path* on a Riemann surface X is a continuous and piecewise C^∞ function $\gamma : [a, b] \to X$ from a closed interval in $\mathbb{R}$ to X. The points $\gamma(a)$ and $\gamma(b)$ are the *endpoints* of the path ($\gamma(a)$ is sometimes called the *initial point*). We say the path γ is *closed* if $\gamma(a) = \gamma(b)$.

There are several obvious remarks to make.

EXAMPLE 3.2. Let $\gamma : [a, b] \to X$ be a path on X. Suppose that $\alpha : [c, d] \to [a, b]$ is a continuous and piecewise C^∞ function sending c to a and d to b. Then $\gamma \circ \alpha$ is a path on X. This is referred to as a *reparametrization* of the path γ. Any path γ may be reparametrized so that its domain is $[0, 1]$.

EXAMPLE 3.3. Let $\gamma : [a, b] \to X$ be a path on X. The *reversal* of γ, denoted by $-\gamma$, is the path defined by sending $t \in [a, b]$ to $\gamma(a + b - t)$. Its initial point is the endpoint of γ, and its endpoint is the initial point of γ.

EXAMPLE 3.4. If $F : X \to Y$ is a C^∞ map