G. Show that the polynomial $f(z,w) = w^2 - h(z)$ is an irreducible polynomial if and only if $h(z)$ is a polynomial which is not a perfect square. Show that $f(z,w)$ is a nonsingular polynomial if and only if $h(z)$ has distinct roots.

H. Let X be an affine plane curve of degree 2, that is, defined by a quadratic polynomial $f(z,w)$. (Such a curve is called an *affine conic*.) Suppose that $f(z,w)$ is singular. Show that in fact f factors as the product of two linear polynomials, so that X is therefore the union of two intersecting lines. Give an example of a smooth affine plane conic.

I. Give an example of a smooth irreducible affine plane curve of arbitrary degree. Make sure you check the irreducibility!

J. Let ϕ be holomorphic in a neighborhood of $p \in \mathbb{C}$. Assume that $\phi'(p) \neq 0$. Prove (using the Implicit Function Theorem) that there exists a neighborhood U of p such that $\phi|_U$ is a chart on $\mathbb{C}$.

3. Projective Curves

The Projective Line $\mathbb{P}^1$ is the first in a series of examples which encompass the most important and interesting compact Riemann surfaces. These are surfaces which are embedded in *projective space*. We first discuss the case of projective *plane* curves.

The Projective Plane $\mathbb{P}^2$. We will make a construction very similar to that made for the projective line $\mathbb{P}^1$.

DEFINITION 3.1. The *projective plane* $\mathbb{P}^2$ is the set of 1-dimensional subspaces of $\mathbb{C}^3$.

If (x,y,z) is a nonzero vector in $\mathbb{C}^3$, its span is denoted by $[x:y:z]$ and is a point in the projective plane; every point in the projective plane may be written in this way. Note that

$$[x:y:z] = [\lambda x : \lambda y : \lambda z]$$

for any nonzero number λ; indeed, $\mathbb{P}^2$ can be viewed as the quotient space of $\mathbb{C}^3 - \{0\}$ by the multiplicative action of $\mathbb{C}^*$. In this way it inherits a Hausdorff topology, which is the quotient topology coming from the natural map from $\mathbb{C}^3 - \{0\}$ onto $\mathbb{P}^2$.

The entries in the notation $[x:y:z]$ are called the *homogeneous coordinates* of the corresponding point in the projective plane. The homogeneous coordinates are not unique, as noted above; however whether they are zero or not is well defined.

The space $\mathbb{P}^2$ can be covered by the three open sets

$$U_0 = \{[x:y:z] \mid x \neq 0\}; U_1 = \{[x:y:z] \mid y \neq 0\}; U_2 = \{[x:y:z] \mid z \neq 0\}.$$

Each open set U_i is homeomorphic to the affine plane $\mathbb{C}^2$. The homeomorphism on U_0 is given by sending $[x:y:z] \in \mathbb{P}^2$ to $(y/x, z/x) \in \mathbb{C}^2$; its inverse sends

Problems II.1

A. Check that all of the functions of Examples 1.3 through 1.11 are holomorphic as claimed.
B. Check that all of the functions of Examples 1.16 through 1.23 are meromorphic as claimed.
C. Let L be a lattice in $\mathbb{C}$ and let X be the torus $\mathbb{C}/L$. Let $\pi : \mathbb{C} \to X$ be the quotient map. Show that a function f on X is meromorphic if and only if the composition $f\pi$ is a meromorphic function on $\mathbb{C}$.
D. Prove Lemma 1.26.
E. Prove Lemma 1.28.
F. Prove Lemma 1.29.
G. Verify all of the statements of Example 1.30.
H. Prove Liouville's Theorem (that a bounded entire function on $\mathbb{C}$ is constant) by showing that a bounded entire function extends to a holomorphic function on the (compact) Riemann Sphere $\mathbb{C}_\infty$.
I. Prove without invoking the Maximum Modulus Theorem that any rational function which is holomorphic at every point of the Riemann Sphere $\mathbb{C}_\infty$ is in fact a constant.

2. Examples of Meromorphic Functions

Meromorphic Functions on the Riemann Sphere. We have seen in Example 1.18 that any rational function $r(z) = p(z)/q(z)$ is meromorphic on the whole Riemann Sphere. In fact, the converse is true:

THEOREM 2.1. *Any meromorphic function on the Riemann Sphere is a rational function.*

PROOF. Let f be a meromorphic function on the Riemann Sphere $\mathbb{C}_\infty$. Since $\mathbb{C}_\infty$ is compact, it has finitely many zeroes and poles. Let $\{\lambda_i\}$ be the set of zeroes and poles of f in the finite complex plane $\mathbb{C}$, and assume that $\text{ord}_{z=\lambda_i}(f) = e_i$. Consider the rational function

$$r(z) = \prod_i (z - \lambda_i)^{e_i}$$

which has the same zeroes and poles, to the same orders, as f does, in the finite plane (see Example 1.30). Let $g(z) = f/r(z)$; g is a meromorphic function on $\mathbb{C}_\infty$, with no zeroes or poles in the finite plane. Therefore, as a function on $\mathbb{C}$, it is everywhere holomorphic, and has a Taylor series

$$g(z) = \sum_{n=0}^{\infty} c_n z^n$$

a known Riemann surface X and find a suitable map from X into projective space.

Problems II.2

A. Consider the projective line $\mathbb{P}^1$. Fix a point $p \in \mathbb{P}^1$, and a finite set $S \subset \mathbb{P}^1$ with $p \notin S$. Show that there exists a meromorphic function f on $\mathbb{P}^1$ with a simple zero at p and no zeroes or poles at any of the points of S.

B. Show that the series defining the theta-function converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

C. Show that $\theta(z+1) = \theta(z)$ for every z in $\mathbb{C}$.

D. Show that $\theta(z+\tau) = e^{-\pi i[\tau+2z]}\theta(z)$ for every z in $\mathbb{C}$.

E. Show that z_0 is a zero of θ if and only if $z_0 + m + n\tau$ is a zero of θ for every m and n in $\mathbb{Z}$. Moreover the order of zero of θ at z_0 is the same as the order of zero at $z_0 + m + n\tau$.

F. Show that the only zeroes of θ are at the points $(1/2) + (\tau/2) + m + n\tau$, for integers m and n, and that these zeroes are simple. (Hint: integrate θ'/θ around a fundamental parallelogram.)

G. Let $\{p_i\}$ and $\{q_i\}$ be two sets of d points on a complex torus $X = \mathbb{C}/L$ (repetitions are allowed). Show that there exist numbers $\{x_i\}$ and $\{y_i\}$ in $\mathbb{C}$ such that $\pi(x_i) = p_i$ and $\pi(y_i) = q_i$ for every i with $\sum_i x_i = \sum_i y_i$ if and only if $\sum_i p_i = \sum_i q_i$ in the quotient group law of X.

H. Consider the complex torus $X = \mathbb{C}/L$. Fix a point $p \in X$, and a finite set $S \subset X$ with $p \notin S$. Show that there exists a meromorphic function f on X with a simple zero at p and no zeroes or poles at any of the points of S.

3. Holomorphic Maps Between Riemann Surfaces

The Definition of a Holomorphic Map. Modern geometric philosophy holds firmly to the notion that the first thing one does after defining the objects of interest is to define the functions of interest. In our case the objects are Riemann surfaces, and we have already addressed complex-valued functions on Riemann surfaces. However "functions" are to be taken also in the sense of mappings between the objects; once we define such mappings, we will have a *category* of Riemann surfaces.

In the case of Riemann surfaces, which have local complex coordinates, the natural property of a mapping is to be holomorphic. Let X and Y be Riemann surfaces.

DEFINITION 3.1. A mapping $F: X \to Y$ is *holomorphic at* $p \in X$ if and only if there exists charts $\phi_1 : U_1 \to V_1$ on X with $p \in U_1$ and $\phi_2 : U_2 \to V_2$ on Y with $F(p) \in U_2$ such that the composition $\phi_2 \circ F \circ \phi_1^{-1}$ is holomorphic at $\phi_1(p)$. If F is defined on an open set $W \subset X$, then we say F is *holomorphic on* W if F is holomorphic at each point of W. In particular, F is a *holomorphic map* if and only if F is holomorphic on all of X.