D. Show that if G is a group, and X is a topological space in which every pair of nonempty open sets intersect, then the constant presheaf G_X (defined by setting $G_X(U) = G$ for every nonempty open subset $U \subseteq X$) is indeed a sheaf on X.

E. Check that the general skyscraper construction produces a sheaf.

F. Prove that a totally discontinuous sheaf is a skyscraper sheaf if and only if the trivial group is used at all but a discrete set of points of X.

G. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves of abelian groups on X. Define the direct sum sheaf $\mathcal{F} \oplus \mathcal{G}$ by setting

$$\mathcal{F} \oplus \mathcal{G}(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$$

for an open set $U \subseteq X$. Define the restriction maps for $\mathcal{F} \oplus \mathcal{G}$ using the restriction maps of $\mathcal{F}$ and of $\mathcal{G}$, and show that $\mathcal{F} \oplus \mathcal{G}$ is a sheaf on X.

H. Prove that if X is a Riemann surface and Y is an open subset, then

$$\mathcal{O}_X|_Y = \mathcal{O}_Y.$$

In general, show that if Z and Y are both open subsets, with $Z \subset Y \subset X$, and $\mathcal{F}$ is a sheaf on X, then

$$(\mathcal{F}|_Y)|_Z = \mathcal{F}|_Z.$$

I. Let $\mathcal{F}$ be a sheaf on X, and fix a point $p \in X$. Consider the disjoint union set $D = \sqcup_{U \ni p} \mathcal{F}(U)$ (note that the union is only over the open neighborhoods of p). Define an equivalence relation on D as follows: if $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ then we declare $f \sim g$ if there is an open neighborhood W of p contained in $U \cap V$ such that $\rho^U_W(f) = \rho^V_W(g)$. Show that this is an equivalence relation on D. The set of equivalence classes is called the *stalk* of $\mathcal{F}$ at p, and is denoted by $\mathcal{F}_p$. Note that there is a natural map $\pi : \mathcal{F}(U) \to \mathcal{F}_p$ for any open neighborhood U of p, sending a section of $\mathcal{F}$ over U to its equivalence class in the stalk.

J. Show that given any two elements s_1 and s_2 in the stalk $\mathcal{F}_p$, there is an open neighborhood U of p such that both s_i are represented by sections of $\mathcal{F}$ over U.

K. Show that if $\mathcal{F}$ is a sheaf of groups, then the stalk $\mathcal{F}_p$ inherits the group structure in the following way: if s_1 and s_2 are two elements of $\mathcal{F}_p$, then find an open neighborhood U of p and sections f_1 and f_2 in $\mathcal{F}(U)$ representing s_1 and s_2 respectively; then set $s_1 + s_2$ to be the equivalence class of $f_1 + f_2$ in $\mathcal{F}(U)$. Show this is well defined and gives a group structure on the stalk. Show that for any U the map $\pi : \mathcal{F}(U) \to \mathcal{F}_p$ is a homomorphism.

L. Show that the stalks of a locally constant sheaf $\underline{G}$ are all isomorphic to the group G.

M. If X is a Riemann surface, show that the stalks of the sheaf $\mathcal{O}_X$ of holomorphic functions on X are all isomorphic (after a choice of local coordinate) to the ring of convergent power series $\mathbb{C}\{z\}$ in one variable over $\mathbb{C}$.

(b) *Composition with* f_i *carries* $\mathcal{O}_{V_i}$ *isomorphically as a sheaf onto* $\mathcal{R}|_{U_i}$ *for each* i*, in the sense that*

$$- \circ f_i : \mathcal{O}_{V_i}(A) \xrightarrow{\cong} \mathcal{R}(f_i^{-1}(A))$$

is a ring isomorphism for every open $A \subset V_i$.

Then the homeomorphisms f_i *are compatible chart maps on the space* X *and define a complex structure on* X*, making* X *into a Riemann surface.*

PROOF. Since the f_i's are already homeomorphisms, they are certainly chart maps; the only thing to check is the compatibility. Fix two such, say $f_1 : U_1 \to V_1$ and $f_2 : U_2 \to V_2$. Let $W = U_1 \cap U_2$. We must show that $h = f_2 \circ f_1^{-1} : f_1(W) \to f_2(W)$ is holomorphic. The function h is defined on $f_1(W)$, which is a subset of V_1. Therefore to check that it is holomorphic, we must show that it is in $\mathcal{O}_{V_1}(f_1(W))$; by condition (b) above it then suffices to check that after composing with f_1 we obtain a function in $\mathcal{R}(W)$. But $h \circ f_1 = f_2|_W$, which by condition (a) is a section of $\mathcal{R}(W)$. □

Thus we see that a Riemann surface can be defined as a certain type of space with a special sheaf of functions defined on it. It would be hard to argue that this approach is simpler than the original. However this same approach may be generalized to many other categories, and is nowadays the "highbrow" approach to defining lots of geometric categories, including topological manifolds, differentiable manifolds, projective varieties, and schemes. We will not pursue this point of view much further, but it is worth being aware of.

Problems IX.2

A. Prove that the identity map is a sheaf map, and that the composition of two sheaf maps is a sheaf map.

B. Check that the sheaf maps described in Examples 2.5 through 2.11 are indeed onto as claimed.

C. Check that the sequences of sheaf maps described in Examples 2.13 through 2.24 are indeed short exact sequences of sheaves as claimed.

D. Check that a short exact sequence of sheaves is exact at the three possible positions. Check that a sheaf map $\phi : \mathcal{F} \to \mathcal{G}$ is 1-1 if and only if the sequence

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G}$$

is exact at $\mathcal{F}$; check that ϕ is onto if and only if the sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \to 0$$

is exact at $\mathcal{G}$.

E. Suppose that a global $\mathcal{C}^\infty$ 2-form η is given on a Riemann surface X. Show that multiplication by η is a sheaf map from the sheaf of $\mathcal{C}^\infty$ functions $\mathcal{C}^\infty_X$ on X to the sheaf of $\mathcal{C}^\infty$ 2-forms $\mathcal{E}^2_X$ on X. Show that this multiplication map is an isomorphism if and only if η is nowhere zero.