$\mathcal{U}$. Then for any n, the diagram

$$\begin{array}{ccc} \check{H}^n(\mathcal{U},\mathcal{F}) & \xrightarrow{\phi_*} & \check{H}^n(\mathcal{U},\mathcal{G}) \\ H^{\mathcal{U}}_{\mathcal{V}} \downarrow & & \downarrow H^{\mathcal{U}}_{\mathcal{V}} \\ \check{H}^n(\mathcal{V},\mathcal{F}) & \xrightarrow{\phi_*} & \check{H}^n(\mathcal{V},\mathcal{G}) \end{array}$$

commutes.

K. Show that if a family of subgroups $\{G_a\}$ of a fixed group G is given, with the property that any two are both contained in a third, then the union $L = \cup_a G_a$ is a subgroup of G, which satisfies the universal property for the direct limit of the subgroups. (Here the maps between the subgroups are the inclusion maps when one is contained in another.)

L. Show that if a direct system of groups $\{G_a\}$ and maps H^b_a are given, such that every map H^b_a is an isomorphism, then the direct limit L of the system of groups is also isomorphic to each, and in fact the natural map $h_a : G_a \to L$ is an isomorphism. (Use the universal property of the direct limit.)

M. Let X be the Riemann Sphere $\mathbb{C}_\infty$, and let $U_0 = X - \{0\}$ and $U_1 = X - \{\infty\}$ be the standard open covering $\mathcal{U}$ of X. Compute $\check{H}^1(\mathcal{U}, \mathcal{O}_X[n \cdot \infty])$ for all n explicitly by writing down the spaces of relevant cochains, computing the 1-cocycles and 1-coboundaries, and taking the quotient group. Show that this cohomology group is a complex vector space.

N. Let $(f_{i_0,\dots,i_n})$ be an n-cocycle for a sheaf $\mathcal{F}$. Show that if any two of the indices are equal, then $f_{i_0,\dots,i_n} = 0$. Show that if all of the indices are distinct, and σ is a permutation of the indices, then $f_{\sigma(i_0),\dots,\sigma(i_n)} = \text{sign}(\sigma) f_{i_0,\dots,i_n}$.

4. Cohomology Computations

As mentioned in the previous sections of this chapter, most of the time one is primarily interested in computing the group of global functions or forms satisfying some local conditions. Cohomologically speaking, this is always some $\check{H}^0$ of a sheaf on the space in question.

Short exact sequences of sheaves give precise relationships between different sheaves, and the computation of global sections can, by appealing to the long exact sequence in cohomology, often be reduced to some computation of an $\check{H}^1$. These in turn can be related to $\check{H}^2$'s, etc. So eventually all the cohomology groups can get involved.

It is most useful to have general statements that with certain sheaves or types of sheaves, higher cohomology groups automatically vanish. If so, then whenever such sheaves appear in a short exact sequence of sheaves, we will have that every third term of the long exact sequence will vanish, which is great information relating the cohomology of the other two sheaves.

We will begin this section by proving that the higher cohomology