PROPOSITION 1.13. *Two complex tori X_τ and $X_{\tau'}$ are isomorphic if and only if there is a matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *in* $\mathrm{SL}_2(\mathbb{Z})$ *such that* $\tau = (a\tau' + b)/(c\tau' + d)$.

The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane $\mathbb{H}$ (the matrix sends τ to $(a\tau + b)/(c\tau + d)$) and so we see that isomorphism classes of complex tori are in 1-1 correspondence with points of the orbit space $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. This orbit space is in fact isomorphic to the complex numbers, via the so-called j-function. The interested reader should consult [Serre73] or [Lang87] for rather complete treatments. But in any case we see that there are uncountably many isomorphism classes of complex tori, and that they vary with essentially one parameter (the lattice generator τ).

Problems III.1

A. Verify that the isomorphism T between two conics described in the text is indeed a holomorphic map. Verify that the map from $\mathbb{P}^1$ to the conic $xz = y^2$ sending $[r : s]$ to $[r^2 : rs : s^2]$ is a holomorphic map.

B. Check that the charts on the glueing space $Z = X \coprod Y/\phi$ defined in the proof of Proposition 1.6 are pairwise compatible.

C. Show that if one glues together $\mathbb{C}$ and $\mathbb{C}$ along $\mathbb{C}^*$ and $\mathbb{C}^*$ via the glueing map $\phi(z) = z$, the resulting space is not Hausdorff.

D. Let $h(x)$ be a polynomial of degree $2g+1+\epsilon$ (with $\epsilon \in \{0, 1\}$) having distinct roots and let $U = \{(x, y) \in \mathbb{C}^2 \mid y^2 = h(x) \text{ and } x \neq 0\}$. As in the text let $k(z) = z^{2g+2}h(1/z)$ and let $V = \{(z, w) \in \mathbb{C}^2 \mid w^2 = k(z) \text{ and } z \neq 0\}$. Show that the mapping $\phi : U \to V$ defined by $(z, w) = (1/x, y/x^{g+1})$ is an isomorphism of Riemann surfaces.

E. Check that the function r defined in the proof of Lemma 1.9 is meromorphic.

F. Let X be the compact hyperelliptic curve defined by $x^2 = 3 + 10t^4 + 3t^8$. Let Y be the compact hyperelliptic curve defined by $w^2 = z^6 - 1$. Let U and V be the corresponding affine plane curves, which are the complements in X and Y respectively of the points at infinity. Show that the function $F : U \to V$ defined by $z = (1 + t^2)/(1 - t^2)$ and $w = 2tx/(1 - t^2)^3$ extends to a holomorphic map from X to Y of degree 2, which is nowhere ramified. What is the genus of X and of Y?

G. Let X be a complex torus. Show that any translation map of X, which is induced from a translation in the complex plane, is a holomorphic map.

H. Let X be a complex torus. Show that the full group of automorphisms of X is a semidirect product of the group of translations with the group $\mathrm{Aut}_0(X)$ of automorphisms fixing 0.

I. Let X be a complex torus, and let F be a nontrivial automorphism of X. Show that if F is not a translation, then F has a fixed point.

J. Let X and Y be complex tori defined by lattices L and M respectively, and $F : X \to Y$ be a holomorphic map induced by a linear map $G(z) = \gamma z + a$ with $\gamma L \subset M$. Show that the degree of F is the index of γL inside M.