f olom. e biunivoca $\Rightarrow f^{-1}$ olomorfa
infatti se f' è annullata in z_0, f non può essere iniettiva in nessun intorno di z_0

Problems I.1

A. Let $\phi_i : U_i \to V_i$, $i = 1, 2$, be complex charts on X with $U_1 \cap U_2 \neq \emptyset$. Suppose that $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is holomorphic. Show that it is bijective, with inverse $\phi_1 \circ \phi_2^{-1} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2)$, proving that $\phi_1 \circ \phi_2^{-1}$ is also holomorphic.

B. Let $\phi : U \to V$ be a complex chart on X, and let $\psi : V \to W$ be a holomorphic bijection between two open sets in $\mathbb{C}$. Show that $\psi \circ \phi : U \to W$ is a complex chart on X. Show that $\psi \circ \phi$ is compatible with any chart on X which is compatible with ϕ.

C. Verify that any two sub-charts of a complex chart are compatible (Example 1.9 of the text).

D. Verify that any two charts in Example 1.2 are compatible.

E. Verify that any two charts in Example 1.3 are compatible.

F. Check that no chart of Example 1.2 is compatible with any chart of Example 1.3 of the notes if their domains intersect.

G. In Example 1.13, where an atlas of the Riemann Sphere is defined, check that indeed $\phi_2 \circ \phi_1^{-1}$ sends z to $1/z$ as stated.

H. Show that equivalence of complex atlases is an equivalence relation.

I. Equivalent atlases may be partially ordered by inclusion. Show that any atlas is equivalent to a unique maximal atlas.

J. Show that holomorphic bijections between open sets in the complex plane preserve the local orientation.

2. First Examples of Riemann Surfaces

In this section we'll present some easy examples of Riemann surfaces, especially of compact Riemann surfaces. These include the projective line, complex tori, and smooth plane curves.

A Remark on Defining Riemann Surfaces. To define a Riemann surface, it would appear that one needs to start with a topological space X, second countable, connected and Hausdorff, and then define a complex atlas on it; in other words, one needs to have the topology first, and then one can impose the complex structure. This is not completely accurate; one can often use the data defining an atlas to also define the topology.

This observation is based on the following remark: if an open cover $\{U_\alpha\}$ of a topological space X is given, then a subset $U \subset X$ is open in X if and only if each intersection $U \cap U_\alpha$ is open in U_α.

More generally, if any collection $\{U_\alpha\}$ of subsets of a set X is given, and topologies are given for each subset U_α, then one can define a topology on X by declaring a set U to be open if and only if each intersection $U \cap U_\alpha$ is open in U_α.

The proo[f] ... requires some of the machinery ... proof here; see [**Shafarevich77**], for example. ... smooth irreducible affine plane curve is a Riemann surface.

EXAMPLE 2.4. Let $h(z)$ be a polynomial in one variable which is not a perfect square. Then the polynomial $f(z,w) = w^2 - h(z)$ is irreducible. Moreover, if $h(z)$ has distinct roots, then f is nonsingular, and its locus of roots X is a Riemann surface. (Prove this for yourself: Problem G below.)

A slight generalization will be useful later. If $f(z,w)$ is an irreducible polynomial, then the points on its locus of roots X where f is singular forms a finite set. (This is nontrivial! But let's go on.) If we delete these points, then the resulting open subset of X is a Riemann surface, using the same charts as given above. This is referred to as the *smooth part* of the affine plane curve X, and in general, if f is an irreducible polynomial, the smooth part of its zero locus is a Riemann surface.

No affine plane curve is compact: as a subset of $\mathbb{C}^2 = \mathbb{R}^4$, it is not a bounded set, since for any fixed z_0, there will be roots w to the polynomial $f(z_0, w) = 0$.

Problems I.2

A. Verify that if any collection of subsets $\{U_\alpha\}$ of a set X are given, and topologies are given for each subset U_α, then a topology can be defined on X by declaring that a subset $U \subseteq X$ is open in X if and only if $U \cap U_\alpha$ is open in U_α for every α.

B. Suppose, in problem A, that each U_α is connected. Form a graph with one vertex (called v_α) for each U_α, and with vertex v_α connected by an edge to v_β if and only if $U_\alpha \cap U_\beta \neq \emptyset$. Prove or disprove: X is connected if and only if the graph is connected.

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C. Check that the function from $\mathbb{P}^1$ to S^2 sending $[z:w]$ to

$$(2\,\mathrm{Re}(w\bar{z}), 2\,\mathrm{Im}(w\bar{z}), |w|^2 - |z|^2)/(|w|^2 + |z|^2)$$

is a homeomorphism onto the unit sphere in $\mathbb{R}^3$. Therefore the projective line is a compact Riemann surface of genus zero.

D. Show that any lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$ with ω_1 and ω_2 linearly independent over $\mathbb{R}$ is a discrete subset of $\mathbb{C}$.

E. Show that a complex torus has topological genus one by constructing an explicit homeomorphism to the product $S^1 \times S^1$ of two circles.

F. Show that the group law of a complex torus X is divisible: for any point $p \in X$ and any integer $n \geq 1$ there is a point $q \in X$ with $n \cdot q = p$. Indeed, show that there are exactly n^2 such points q.