The Banach fixed point theorem (Teorema di Banach-Caccioppoli)

Let (M, d) be a metric space, $\phi : M \to M$ and $z \in M$.

a) z is called a fixed point of ϕ if $\phi(z) = z$.

b) ϕ is called a strict contraction if

$$\exists\, L \in [0,1) \quad \forall\, x, y \in M : \qquad d(\phi(x), \phi(y)) \leqslant L\, d(x, y).$$

Theorem. *Let (M, d) be complete and ϕ a strict contraction. Then ϕ has a unique fixed point. If $x_0 \in M$ is an arbitrary element of M and one defines (recursively) the sequence*

$$x_n = \phi(x_{n-1}), \qquad n = 1, 2, 3, \ldots,$$

then

$$x_n \xrightarrow{n \to +\infty} z.$$

Moreover, the following error estimates are valid:

a) A-priori estimate: $\quad d(x_n, z) \leqslant \dfrac{L^n}{1-L} d(x_1, x_0)$ *for every n.*

b) A-posteriori estimate: $\quad d(x_n, z) \leqslant \dfrac{L}{1-L} d(x_n, x_{n-1})$ *for every n.*

Proof: There is at most one fixed point, since if $z = \phi(z)$ and $z' = \phi(z')$ then

$$d(z, z') = d(\phi(z), \phi(z')) \leqslant L\, d(z, z'),$$

hence $d(z, z') = 0$, i.e., $z = z'$. For the existence note that

$$\begin{aligned} d(x_{k+1}, x_k) &= d(\phi(x_k), \phi(x_{k-1})) \\ &\leqslant L\, d(x_k, x_{k-1}) = L d(\phi(x_{k-1}), \phi(x_{k-2})) \\ &\leqslant L^2\, d(x_{k-1}, x_{k-2}) \leqslant \ldots \leqslant L^k d(x_1, x_0). \end{aligned}$$

Then, for every indices $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n) \\ &\leqslant \sum_{\ell=n}^{m-1} L^\ell\, d(x_1, x_0) \leqslant L^n \sum_{\ell=0}^{+\infty} L^\ell\, d(x_1, x_0) = \frac{L^n}{1-L}\, d(x_1, x_0). \end{aligned}$$

Since $L^n \xrightarrow{n \to +\infty} 0$ this shows that (x_n) is a Cauchy-sequence in M. Since M is complete, the sequence converges; call z its limit. Then, using the continuity of ϕ,

$$z \xleftarrow{n \to +\infty} x_{n+1} = \phi(x_n) \xrightarrow{n \to +\infty} \phi(z)$$

shows that z is a fixed point of ϕ. Moreover,

$$d(z, x_n) \xleftarrow{m\to+\infty} d(x_m, x_n) \leqslant \frac{L^n}{1-L}\, d(x_1, x_0)$$

shows the a-priori estimate. Similarly one finds the a-posteriori estimate: First one has

$$d(x_m, x_n) \leqslant \sum_{\ell=0}^{m-n-1} d(x_{n+\ell+1}, x_{n+\ell}) \leqslant \sum_{\ell=0}^{m-n-1} L^{\ell+1}\, d(x_n, x_{n-1}).$$

Passing to the limit $m \to +\infty$ yields

$$d(z, x_n) \leqslant \sum_{\ell=0}^{+\infty} L^{\ell+1}\, d(x_n, x_{n-1}) = \frac{L}{L+1}\, d(x_n, x_{n-1}).$$

This completes the proof. ∎

An application: Initial value problems

Let $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function satisfying the Lipschitz condition

$$\boxed{\|f(t, x) - f(t, x')\| \leqslant C\|x - x'\| \qquad \forall\, t \in [a, b] \quad \forall\, x, x' \in \mathbb{R}^n,}$$

with some constant $C \geqslant 0$. Let $x_0 \in \mathbb{R}^n$ and $t_0 \in [a, b]$ be fixed.

Picard-Lindelöf Theorem (global version). *With the previous notation, there exists a unique solution $x \in C^1([a, b], \mathbb{R}^n)$ of the initial value problem*

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1}$$

Proof: By the main theorem of calculus, being a continuously differentiable solution of (1) is equivalent to being a continuous solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{2}$$

Now let $M := C([a, b], \mathbb{R}^n)$ and define a metric on M by

$$\mathbf{d}(g, h) = \max_{a \leqslant t \leqslant b} e^{-(C+1)|t-t_0|}\|g(t) - h(t)\|.$$

Recall that the standard metric on M is given by

$$d(g, h) = \max_{a \leqslant t \leqslant b} \|g(t) - h(t)\|.$$

Since

$$e^{-(C+1)(b-a)} \leqslant e^{-(C+1)|t-t_0|} \leqslant 1 \qquad \forall\, t \in [a, b],$$

d and $\mathbf{d}$ are equivalent metrics on M. Hence $(M, \mathbf{d})$ is a complete metric space.

Now we consider the map $\phi : M \to M$ defined by

$$[\phi(g)](t) = x_0 + \int_{t_0}^{t} f(s, g(s))\, ds, \qquad t \in [a, b]. \tag{3}$$

We shall show below that ϕ is a strict contraction on $(M, \mathbf{d})$ with constant $L = \frac{C}{C+1} < 1$. Hence there exists a unique fixed point of ϕ in M. This fixed point is the unique solution of (2) and thus of (1).
Let us assume for simplicity that $t_0 = a$ (the general case works analogously); thus $|t - t_0| = t - a$ for all $t \in [a, b]$. Then

$$\begin{aligned}
|[\phi(g)](t) - [\phi(h)](t)| &= \Big| \int_a^t f(s, g(s)) - f(s, h(s))\, ds \Big| \\
&\leqslant C \int_a^t |g(s)) - h(s)|\, ds \\
&= C \int_a^t e^{(C+1)(s-a)} e^{-(C+1)(s-a)} |g(s)) - h(s)|\, ds \\
&\leqslant C\mathbf{d}(g, h) \int_a^t e^{(C+1)(s-a)}\, ds \\
&\leqslant C\mathbf{d}(g, h) \frac{1}{C+1} e^{(C+1)(s-a)} \Big|_{s=a}^{s=t} \\
&= \frac{C}{C+1} \mathbf{d}(g, h) \Big(e^{(C+1)(t-a)} - 1 \Big) \\
&\leqslant \frac{C}{C+1} \mathbf{d}(g, h) e^{(C+1)(t-a)}.
\end{aligned}$$

Multiplying from the left with $e^{-(C+1)(t-a)}$ and the passing to the maximum over $t \in [a, b]$ yields

$$\mathbf{d}(\phi(g), \phi(h)) \leqslant \frac{C}{C+1} \mathbf{d}(g, h)$$

for arbitrary $g, h \in M$. ■

Example. Let us consider the initial value problem

$$x'(t) = x(t), \qquad x(0) = 1.$$

Here $n = 1$ and $f(t, x) = x$. Obviously f satisfies the Lipschitz condition on every interval $[-a, a]$ with constant $C = 1$. The map ϕ from (3) becomes

$$[\phi(g)](t) = 1 + \int_0^t x(s)\, ds.$$

Let us calculate the sequence of functions (g_n) defined by

$$g_0 \equiv 1, \quad g_n = \phi(g_{n-1}).$$

We have

$$
\begin{aligned}
g_1(t) &= 1 + \int_0^t 1\,ds = 1 + t,\\
g_2(t) &= 1 + \int_0^t 1 + s\,ds = 1 + t + \frac{t^2}{2},\\
&\vdots\\
g_n(t) &= 1 + \int_0^t g_{n-1}(s)\,ds = 1 + t + \frac{t^2}{2} + \ldots + \frac{t^n}{n!}.
\end{aligned}
$$

Therefore

$$g_n(t) = \sum_{k=0}^{n} \frac{t^n}{n!} \xrightarrow{n \to +\infty} \sum_{k=0}^{+\infty} \frac{t^n}{n!} = e^t.$$

Hence $x(t) = e^t$ is the unique (on $\mathbb{R}$) solution of the initial value problem.