M.Sc. in Stochastics and Data Science
Analysis Course A (Advanced)

Homework 4

Exercise 1. Let $T \in \mathscr{D}'(\Omega)$ and $\beta \in \mathbb{N}_0^n$. Define $\partial^\beta T : \mathscr{D}(\Omega) \to \mathbb{C}$ by

$$(\partial^\beta T)(\phi) = (-1)^{|\beta|} T(\partial^\beta \phi), \qquad \phi \in \mathscr{D}(\Omega).$$

Show that $\partial^\beta T$ is a distribution in $\mathscr{D}'(\Omega)$.

Exercise 2. Let $a \in \mathscr{C}^\infty(\mathbb{R})$ e $T \in \mathscr{D}'(\mathbb{R})$. Show that

$$(aT)' = a'\,T + a\,T'$$

as distributions.

Hint: Apply both sides to an arbitrary test function ϕ.

Exercise 3. Let $g(x) = e^x$ and $\mathbb{R}_+ = (0, +\infty)$. Find a linear map

$$T \mapsto T \circ g : \mathscr{D}'(\mathbb{R}_+) \to \mathscr{D}'(\mathbb{R})$$

such that $T_f \circ g = T_{f \circ g}$ for every regular distribution $T_f \in \mathscr{D}'(\mathbb{R}_+)$.

Hint: Show that one can write $T_{f\circ g}(\phi)$ in the form $T_f(A(\phi))$ with a suitable operator A.

Exercise 4. Prove Theorem 1.14.

Hint: Proceed similarly to Example 1.13, writing $\displaystyle\int_{-\infty}^{+\infty}$ as the sum of $\displaystyle\int_{-\infty}^{x_0}$ and $\displaystyle\int_{x_0}^{+\infty}$.

Exercise 5. On $\mathbb{R}$ consider the first order differential operator A defined by

$$Ay(x) = ay'(x) + by(x) \qquad (a, b \text{ arbitrary fixed numbers with } a \neq 0).$$

Let $v \in \mathscr{C}^\infty(\mathbb{R})$ be the unique function with $av' + bv = 0$ on $\mathbb{R}$ and $v(0) = 1/a$. Use Theorem 1.14 for showing that the regular distribution T_u with density

$$u(x) = \begin{cases} v(x) & : x > 0 \\ 0 & : x < 0 \end{cases}$$

is a fundamental solution of A, i.e., $AT_u = \delta$. Determine $v(x)$ explicitely.