M.Sc. in Stochastics and Data Science
Analysis Course A (Advanced)

Homework 5

Exercise 1. Let $f(x) = e^{-\|x\|^2/2}$ (note that $f \in \mathscr{S}(\mathbb{R}^n)$). Show that $\widehat{f} = (2\pi)^{n/2} f$.

Hint: Observe that

$$e^{-\|x\|^2/2} = e^{-x_1^2/2} \cdot \ldots \cdot e^{-x_n^2/2}, \qquad e^{-ix\xi} = e^{-ix_1\xi_1} \cdot \ldots \cdot e^{-ix_n\xi_n}.$$

Exercise 2. Recall that for $f \in L^2(\mathbb{R}^n)$ the Fourier transform is defined as

$$\mathscr{F} f = \lim_{k \to +\infty} \widehat{\varphi}_k \qquad (\text{in } L^2(\mathbb{R}^n)),$$

where $(\varphi_k) \subset \mathscr{S}(\mathbb{R}^n)$ is a sequence such that $\varphi_k \to f$ in $L^2(\mathbb{R}^n)$. Show that this definition does not depend on the choice of the sequence (φ_k).

Hint: Prove that, for any other sequence $(\psi_k) \subset \mathscr{S}(\mathbb{R}^n)$ such that $\psi_k \to f$ in $L^2(\mathbb{R}^n)$, the limits F and G in $L^2(\mathbb{R}^n)$ of the two sequences $(\widehat{\varphi}_k)$ and $(\widehat{\psi}_k)$ coincide.

Exercise 3. Let $f \in L^2(\mathbb{R})$ and T_f be the associated regular tempered distribution. Show that $\mathscr{F}(T_f) = T_{\mathscr{F} f}$, where on the left-hand side is used the Fourier transform of tempered distributions while on the right-hand side the Fourier transform of L^2-functions.

Exercise 4. Let $T \in \mathscr{S}'(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Let $p_\alpha(x) = x^\alpha$. Show that

$$\widehat{\partial^\alpha T} = i^{|\alpha|} p_\alpha \widehat{T}, \qquad (-i)^{|\alpha|} \widehat{p_\alpha T} = \partial_\xi^\alpha \widehat{T}.$$

Hint: Apply all terms to a test function ϕ. Use the definition of derivative of distributions as well as multiplication of distributions with functions of tempered growth and the corresponding rules for rapidly decreasing functions.