Lecture 1

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VECTOR SPACE (LINEAR SPACE)

$\mathbb{F} \overset{\text{def}}{:=} \mathbb{R}$ or $\mathbb{C}$ (real or complex numbers)

Def. A set $V \neq \emptyset$ (empty set) is a vector space over $\mathbb{F}$ with

1) $+ : V \times V \to V$, $(x,y) \mapsto x+y$ "vector addition"
2) $\cdot : \mathbb{F} \times V \to V$, $(\alpha, x) \mapsto \alpha \cdot x$ "scalar multiplication"

such that:

$$\left.\begin{array}{l} \cdot\ (x+y)+z = x+(y+z) \\ \cdot\ x+y = y+x \\ \cdot\ \alpha\cdot(x+y) = \alpha\cdot x + \alpha y \\ \cdot\ (\alpha+\beta)\cdot x = \alpha\cdot x + \beta\cdot x \\ \cdot\ 1\cdot x = x \\ \cdot\ (\alpha\beta)\cdot x = \alpha\cdot(\beta\cdot x) \end{array}\right\} \quad \forall x,y,z \in V \quad \forall \alpha,\beta \in \mathbb{F}$$

$\forall x,y,z \in V$ ← "vectors"

$\forall \alpha,\beta \in \mathbb{F}$ ← "scalars"

and

- $\exists!\ 0 \in V$, $\forall x \in V$: $0 + x = x$ "zero vector"
- $\forall x \in V$ $\exists!\ \tilde{x} \in V$: $x + \tilde{x} = 0$ "inverse vector"

Def: $-x := \tilde{x}$

Examples. • $V = \mathbb{F}^n = \underbrace{\mathbb{F} \times \cdots \times \mathbb{F}}_{n \text{ times}}$ $\quad (V = \mathbb{R}^n$ or $V = \mathbb{C}^n)$

- V = set of polynomials of degree $\leq n$, $n \in \mathbb{N}$.
- V = set of polynomials of arbitrary degree
- $V = C_{\mathbb{F}}([a,b]) = \{ f : [a,b] \to \mathbb{F}$ continuous on $[a,b] \}$

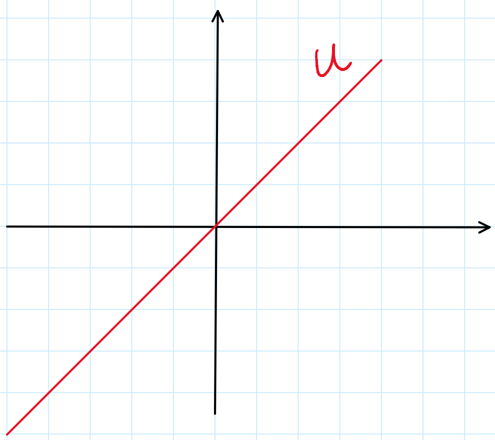
Def. $U \subseteq V$ (vector space) is called a **subspace** of V if "$+$" and "$\cdot$" of V make U a vector space.

Thrm. V vector space, $U \subseteq V$. Then

U is a subspace of $V \iff \forall \alpha, \beta \in \mathbb{F}, \forall x, y \in U$
$\alpha \cdot x + \beta \cdot y \in U$

Write αx in place of $\alpha \cdot x$ for brevity

Example. $V = \mathbb{R}^2$ $\quad U = \{ (x, x) \mid x \in \mathbb{R} \}$



Note: $\underset{\substack{\uparrow \\ \text{scalar}}}{0} x = \underset{\substack{\uparrow \\ \text{vector}}}{0}$, $\quad \forall x \in V$

$\Rightarrow \quad \{0\}$ smallest subspace of V

Def. A linear combination of $x_1, \dots, x_n \in V$ is any vector $x \in V$ of the form

$$x = \sum_{j=1}^{n} \alpha_j x_j, \qquad \alpha_j \in \mathbb{F}, \quad j = 1, \dots, n$$

$\{v_1, \dots, v_n\} \subset V$ is called linearly independent if

$$\left.\begin{array}{l} \alpha_1, \dots, \alpha_n \\ \alpha_1 v_1 + \dots + \alpha_n v_n = 0 \end{array}\right\} \Rightarrow \alpha_1 = \dots = \alpha_n = 0$$

Otherwise, $\{v_1, \dots, v_n\}$ is called linearly dependent

Def. $\phi \neq A \subseteq V$ subset of V.

$$\mathrm{Span}(A) := \left\{ \sum_{j=1}^{n} \alpha_j x_j \;\middle|\; n \in \mathbb{N},\ \alpha_j \in \mathbb{F},\ x_j \in A \right\}$$

= set of all possible linear combinations of elements from A.

$\mathrm{Span}(A)$ is a subspace of V, the smallest subspace of V containing A.

Def. A basis of V is a set $\{v_1, \dots, v_n\} \subset V$ such that:

1) $\{v_1, \dots, v_n\}$ linearly independent
2) $\mathrm{Span}(\{v_1, \dots, v_n\}) = V$

Note: a basis may not exist

Thm. $\{v_1, \dots, v_n\}$ basis of V. Then:

1) Any other basis has exactly n elements

2) $\{w_1, \dots, w_m\} \subset V$ lin. independent $\Rightarrow m \leq n$

3) Any $x \in V$ satisfies $x = \sum_{j=1}^{n} \alpha_j v_j$ for a unique choice of scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$.

Def. If V has a basis define

$\dim V := \#$ elements of a basis

$\uparrow$ dimension of V $\qquad$ $\uparrow$ number

If V does not have a basis we write

$\dim V = \infty \quad (+\infty)$.

Example $\dim \mathbb{F}^n = n$. "standard basis" is

$e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots,$

$e_n = (0, 0, \dots, 0, 1)$

Example $S \neq \emptyset$ set, V vector space over $\mathbb{F}$.

$$F(S, V) := \{ \text{functions } f : S \to V \}$$

is a vector space with:

- "+" $\quad (f_1 + f_2)(s) = f_1(s) + f_2(s), \ \forall s \in S$
- $\alpha \cdot f$ defined by $(\alpha \cdot f)(s) = \alpha \cdot f(s), \ \forall \alpha \in \mathbb{F}, \ \forall s \in S$

$f_1, f_2, f \in F(S, V)$

check as exercise that $F(S, V)$ is a vector space

Theorem. $\dim F(S, V) = \# S \cdot \dim V$

(where $n \cdot \infty = \infty \cdot n = \infty$)

Proof. Only the case of $V=\mathbb{R}$

a) $S=\{s_1,\dots,s_n\}$ finite set, $\#S=n$

$$f_j(s):=\begin{cases}1, & s=s_j\\ 0 & \text{otherwise}\end{cases} \qquad j=1,\dots,n$$

$$f_j\in F(S,V)$$

a.1) $\{f_1,\dots,f_n\}\subset F(S,V)$ is linearly independent

$$\sum_{j=1}^{n}\alpha_j f_j=0$$

$$0=\left(\sum_{j=1}^{n}\alpha_j f_j\right)(s_k)=\sum_{j=1}^{n}\alpha_j \underbrace{f_j(s_k)}_{=\delta_{jk}}=\alpha_k$$

$$\delta_{jk}:=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases}$$

So $\alpha_k=0$, $\forall k=1,\dots,n$

a.2) $F(S,V)=\mathrm{Span}(\{f_1,\dots,f_n\})$

Let $f\in F(S,V)$ define $\alpha_j:=f(s_j)$

$$\left(\sum_{j=1}^{n}\alpha_j f_j\right)(s_k)=\sum_{j=1}^{n}\alpha_j \underbrace{f_j(s_k)}_{\delta_{jk}}=\alpha_k=f(s_k)$$

$\forall k=1,\dots,n$

$\Rightarrow$ $\{f_1,\dots,f_n\}$ is a basis for $F(S,V)$

$\Rightarrow \dim F(S,V)=n$

b) $\#S = \infty$. Consider $\{x_1, x_2, x_3, \dots\} \subset S$ and define the functions f_i as above

$\{f_1, \dots, f_n\}$ lin. independent as above, for every choice of $n \in \mathbb{N}$.

$\Rightarrow$ by the previous part $\Rightarrow \dim F(S, V) = \infty$

Ex 1. $F([a,b], \mathbb{R}) = \{ f : [a,b] \to \mathbb{R} \}$

Ex 2. $S = \{1, 2, \dots, n\}$ $\quad V = \mathbb{R}$

$F(S, \mathbb{R}) = \mathbb{R}^n$ "can be identified"

$$f : S \to \mathbb{R}$$
$$1 \mapsto v_1$$
$$2 \mapsto v_2 \qquad\qquad f \simeq (v_1, \dots, v_n) \in \mathbb{R}^n$$
$$\vdots$$
$$n \mapsto v_n$$

LINEAR OPERATORS

Def Consider V, W vector spaces over $\mathbb{F}$. A map $T : V \to W$ is called *linear* if

$$T(\underbrace{\alpha x + \beta y}_{\text{operations in } V}) = \underbrace{\alpha T(x) + \beta T(y)}_{\text{operations in } W}, \quad \forall \alpha, \beta \in \mathbb{F}, \quad \forall x, y \in V$$

$L(V, W) :=$ set of all linear $T : V \to W$

$= \{ T : V \to W, \ T \text{ linear} \} \subseteq F(V, W)$

Thrm. $L(V, W)$ is a subspace of $F(V, W)$.

Proof. Consider $T_1, T_2 \in L(V, W)$, $\mu_1, \mu_2 \in \mathbb{F}$

check that $\mu_1 T_1 + \mu_2 T_2 \in L(V, W)$

$$(\mu_1 T_1 + \mu_2 T_2)(\alpha x + \beta y) = \dots \overset{?}{=} \alpha(\mu_1 T_1 + \mu_2 T_2)(x) + \beta(\mu_1 T_1 + \mu_2 T_2)(y)$$

$+\beta(\mu_1 T_1 + \mu_2 T_2)(g)$

Example. $\cdot$ $T \in L(\mathbb{R}, \mathbb{R}) \iff T(x) = \alpha x$, $\alpha \in \mathbb{F}$

- $V = W = \mathbb{F}^n$, A $n \times n$ matrix,

  $T(x) := Ax \Rightarrow T$ is a linear operator

  $T : \mathbb{F}^n \to \mathbb{F}^n$, that is $T \in L(\mathbb{F}^n, \mathbb{F}^n)$

Thrm. V, W vector spaces, $U \subseteq V$, $Z \subseteq W$ subspaces. $T \in L(V, W)$. Then

1) $T(U) := \{T(x),\ x \in U\}$ subspace of W

2) $T^{-1}(Z) = \{x \in V : T(x) \in Z\}$ subspace of V

In particular,

$\operatorname{Im} T := T(V)$ image / range of T

$\operatorname{Ker} T := T^{-1}(\{0\}) = \{x \in V \mid T(x) = 0\}$

Kernel of T

Thrm. $T \in L(V, W)$, then

$$\dim \operatorname{Ker} T + \dim \operatorname{Im} T = \dim V$$

(where $n + \infty = \infty + n = \infty$)

Recall that $f \in F(S, V)$ is called one-to-one or injective if for all $x_1, x_2 \in S$ holds:

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Thrm. $T \in L(V, W)$ injective $\iff \operatorname{Ker} T = \{0\}$

Matrix seepresentation of linear operators

V, W finite dimensional vector spaces

V, W finite dimensional vector spaces

$\{v_1, \dots, v_n\} \subset V$ be a basis for V

$\{w_1, \dots, w_m\} \subset W$ be a basis for W

$T \in L(V, W) \Rightarrow \exists$ unique choice $\alpha_{ij} \in \mathbb{F}$

such that

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i \quad , \qquad j = 1, \dots, n$$

unique !

The $(m \times n)$-matrix

$$A = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix}$$

is called the matrix-representation of T with respect to the two basis.