GENERAL FORM OF WAT

(M, d) compact metric space. $C_{\mathbb{R}}(M) = \{ f : M \to \mathbb{R}$ continuous $\}$ $d(f,g) = \max_{x \in M} |f(x) - g(x)|$

Thrm. $(C_{\mathbb{R}}(M), d)$ is a complete metric space.

Def. $\mathcal{A} \subset C_{\mathbb{R}}(M)$ is called unital algebra if

1) $\mathcal{A}$ is a subspace of $C_{\mathbb{R}}(M)$
2) $f, g \in \mathcal{A} \Rightarrow fg \in \mathcal{A}$
3) $f \equiv 1$ belongs to $\mathcal{A}$.

Def. $\mathcal{A}$ separates points of M if

$$\forall x, y \in M : x \neq y \quad \exists f \in \mathcal{A} \mid \quad f(x) \neq f(y)$$

Example. $M = [a, b]$, $\mathcal{A} = \mathcal{P}_{\mathbb{R}}$ is a unital algebra which separates points of M. For instance take $f(x) = x \in \mathcal{P}_{\mathbb{R}}$ and if $x \neq y \Rightarrow f(x) \neq f(y)$.

STONE-WEIERSTRASS THRM (real version).
(M, d) compact metric space, $\mathcal{A} \subset C_{\mathbb{R}}(M)$ a unital algebra that separates points. Then $\mathcal{A}$ is dense in $C_{\mathbb{R}}(M)$.

$$C_{\mathbb{C}}(M) = \{ f : M \to \mathbb{C} \quad \text{continuous} \}$$

Def. $\mathcal{A} \subset C_{\mathbb{C}}(M)$ is closed under conjugation if $f \in \mathcal{A} \Rightarrow \overline{f} \in \mathcal{A}$.

STONE-WEIERSTRASS THRM (complex version)
(M, d) compact metric space, $\mathcal{A} \subset C_{\mathbb{C}}(M)$ a unital algebra that separates points closed under conjugation. Then $\mathcal{A}$ is dense in $C_{\mathbb{C}}(M)$.

SEPARABLE SPACES

Def. A set X is said countable if it is either finite ($\# X < \infty$) or there exists a bijective map $\varphi : \mathbb{N} \to X$. In the latter case, $x_n := \varphi(n)$ and $X = \{x_1, x_2, \dots, x_n, \dots\}$

Def. (M, d) metric space is separable if it contains a dense countable subset.

Examples. $\mathbb{Q}$ is countable and dense in $\mathbb{R}$ so $\mathbb{R}$ is separable. $\mathbb{R}^n$ is separable since $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable. $\mathbb{C}$ is separable since the set $\{ r+qi, \; r, q \in \mathbb{Q} \} \simeq \mathbb{Q}^2$ is countable and dense in $\mathbb{C}$

Theorem. $C_{\mathbb{R}}([a,b])$ is separable.

Proof. $\mathcal{P}^n_{\mathbb{Q}} := \{ a_0 + a_1 x + \dots + a_n x^n \mid a_j \in \mathbb{Q}, \; j = 0, \dots, n \}$ $n \in \mathbb{N}$.

$$\mathcal{P}^n_{\mathbb{Q}} \cong \mathbb{Q}^{n+1} = \underbrace{\mathbb{Q} \times \dots \times \mathbb{Q}}_{n+1 \text{ times}}$$

$\Rightarrow \mathcal{P}^n_{\mathbb{Q}}$ is countable

$\mathcal{P}_{\mathbb{Q}}$: set of polynomials with rational coefficients

$\mathcal{P}_{\mathbb{Q}} = \bigcup_{n \in \mathbb{N}} \mathcal{P}^n_{\mathbb{Q}}$ $\quad \mathcal{P}_{\mathbb{Q}}$ is countable since it's a countable union of countable sets.

Let $\varepsilon > 0$ be given. $\forall f \in C_{\mathbb{R}}([a,b])$ $\exists\, p \in \mathcal{P}_{\mathbb{Q}}$ such that $d(f, p) < \varepsilon$?

By WAT $\exists\, q \in P_{\mathbb{R}} : d(f, q) < \frac{\varepsilon}{2}$

$q(x) = a_0 + a_1 x + \dots + a_N x^N, \quad a_j \in \mathbb{R}, \quad j = 0, \dots, N$

$p(x) := r_0 + r_1 x + \dots + r_N x^N, \quad r_j \in \mathbb{Q}$ such that

$$d(p, q) \leq \max_{x \in [a,b]} \sum_{j=0}^{N} |r_j - a_j| \, |x^j| \leq C \sum_{j=0}^{N} |r_j - a_j|$$

with $C := \max_{x \in [a,b]} \max_{0 \leq j \leq N} |x^j|$

Choose r_j's with

$$|r_j - a_j| < \frac{\varepsilon}{2C(N+1)}, \quad \forall j = 0, \dots, N$$

then $d(p, q) < \frac{\varepsilon}{2}$

So $d(f, q) \leq \underbrace{d(f, q) + d(q, p)}_{\text{Triangle inequality}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. ■

NORMED SPACES

Let X be a vector space over $\mathbb{F}$. A **norm** on X is a map $\|\cdot\| : X \to \mathbb{R}$ such that:

1) $\|x\| \geq 0$
2) $\|x\| = 0 \iff x = 0$
3) $\|\alpha x\| = |\alpha| \, \|x\|$
4) $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)

$\forall x, y \in X, \ \forall \alpha \in \mathbb{F}$

$(X, \|\,\|)$ is called **normed space**. Any $x \in X$ with $\|x\| = 1$ is called "unit vector".

Example. (I) $X = \mathbb{F}^m \qquad \|x\| = \sqrt{|x_1|^2 + \dots + |x_m|^2}, \quad x = (x_1, \dots, x_m)$

standard Euclidean norm.

(II) $X = C_{\mathbb{F}}([a,b])$ with $\|f\| = \max_{x \in [a,b]} |f(x)|$

Proof. 1) and 2) are trivial.

3) $\forall \alpha \in \mathbb{F}$, $\|\alpha f\| = \max_{x\in[a,b]} |\alpha f(x)|$

$$= |\alpha| \max_{x\in[a,b]} |f(x)| = |\alpha| \, \|f\|$$

4) $\forall x \in [a,b]$ $\quad |f(x)+g(x)| \leq |f(x)|+|g(x)|$

$$\leq \underbrace{\|f\| + \|g\|}_{\text{upper bound}}, \ \forall x$$

$$\Rightarrow \|f+g\| \leq \|f\| + \|g\|$$

Exercise 1, HMW 1

L^p SPACES

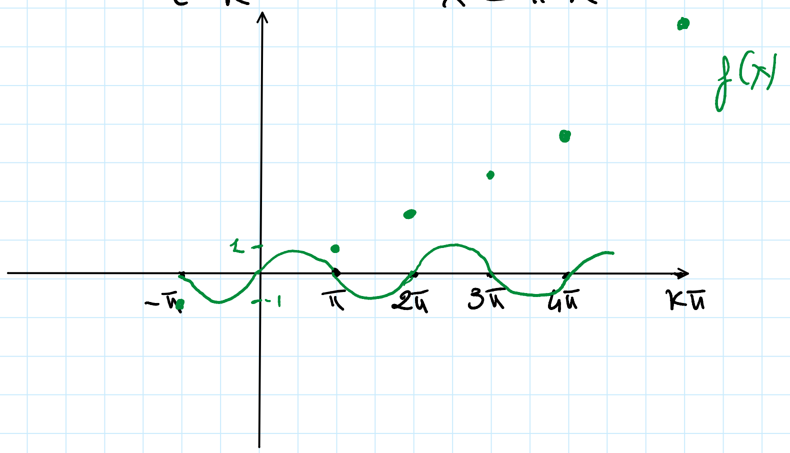
$(\underset{\text{set}}{X}, \underset{\sigma\text{-algebra}}{\Sigma}, \underset{\text{measure}}{\mu})$ measure space

$1 \leq p < \infty$ $\quad \mathcal{L}^p(X) := \{ f : X \to \mathbb{F} \mid f$ measurable and $\int_X |f(x)|^p \, d\mu < \infty \}$

$$\underset{\text{(essential supremum)}}{\operatorname{ess\,sup}} |f| = \inf \{ C > 0 : |f(x)| \leq C \text{ a.e. } x \in X \} = \inf \{ \sup_{X \setminus N} |f(x)| \mid N \subseteq X, \mu(N) = 0 \}$$

$\mathcal{L}^\infty(X) = \{ f : X \to \mathbb{F} \mid f$ measurable and $\operatorname{ess\,sup} |f| < \infty \}$

Example. $f(x) = \begin{cases} \sin x, & x \neq \pi k, \\ k & x = \pi k, \end{cases}$ $\quad k \in \mathbb{Z}$



$\mu(\{\pi k\}_{k\in\mathbb{Z}}) = 0 \quad \Rightarrow \quad \text{ess sup}|f| = 1$

$1 \le p \le \infty \qquad L^p(X) = \mathcal{L}^p(X)/\sim$

where $\sim$ denotes the following equivalent relation:

$$f \sim g \quad \Leftrightarrow \quad f(x) = g(x) \quad \text{a.e. } x \in X$$

Thus the elements of $L^p(X)$ are the equivalent classes $[f] := \{g \in \mathcal{L}^p(X) : f(x) = g(x) \text{ a.e. } x\}$

with addition $\quad [f] + [g] = [f+g]$

and scalar multiplication $\quad \alpha[f] = [\alpha f]$.

CONVENCTION: write f instead of $[f]$ and say "function" instead of "equivalence class of functions".

Proposition. Let X be a measure space such that all open sets U satisfy $\mu(U) > 0$. Then if $f \in C_\#(X)$ we have:

$$\text{ess sup}|f| = \sup|f|.$$

Note. $\int_X |f(x)|^p \, d\mu = 0 \quad \Leftrightarrow \quad |f(x)|^p = 0 \quad \text{a.e. } x \in X$

$\qquad \Leftrightarrow \quad |f(x)| = 0 \quad \text{a.e. } x \in X$

$\qquad \Leftrightarrow \quad f(x) = 0 \quad \text{a.e. } x \in X$

Lemma. $|f(x)| \le \text{ess sup}|f| \quad \text{a.e. } x \in X$

ℓ^p SPACES

$X = \mathbb{N}, \qquad \Sigma = \mathcal{P}(\mathbb{N}) = \text{all subsets of } \mathbb{N}$

power set

$$\mu_c(S) := \begin{cases} \# S, & S \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$$

$\mu_c(\phi) = 0$ by definition

μ_c is a measure called counting measure

Note: ϕ is the only set of measure zero!

$f: \mathbb{N} \to \mathbb{F}$ are sequences denoted by $(a_n)_n$

$\mathcal{L}^p(\mathbb{N}, \mu_c) = L^p(\mathbb{N}, \mu_c)$

every $f: \mathbb{N} \to \mathbb{F}$ is measurable since $\Sigma = \mathcal{P}(\mathbb{N})$

$$\int_{\mathbb{N}} |f(n)|^p d\mu_c = \sum_{n \in \mathbb{N}} |f(n)|^p \qquad \text{write } \sum_{n \in \mathbb{N}} |a_n|^p$$

$a_n = f(n)$.

Definition. $\ell^p(\mathbb{N}) = L^p(\mathbb{N}, \mu_c) = \mathcal{L}^p(\mathbb{N}, \mu_c)$,

$1 \leq p \leq \infty$

This means:

$1 \leq p < \infty$ $\qquad \ell^p(\mathbb{N}) = \{ (a_n)_n \mid \sum_{n \in \mathbb{N}} |a_n|^p < \infty \}$

$p = \infty$ $\qquad \ell^\infty(\mathbb{N}) = \{ (a_n)_n \mid \sup_{n \in \mathbb{N}} |a_n| < \infty \}$

L^p SPACES ARE NORMED SPACES

(X, Σ, μ) measure space.

$$\|f\|_p := \begin{cases} \left(\int_X |f(x)|^p d\mu \right)^{1/p}, & 1 \leq p < \infty \\ \operatorname{ess\,sup} |f|, & p = \infty, \end{cases}$$

defines a norm on $L^p(X)$.

Exercise Show that $\|\cdot\|_\infty$ is a norm on $C_{\mathbb{F}}([a,b])$.

$\mathcal{L}^p(X)$ is a vector space.

Consider $f, g \in \mathcal{L}^p(X)$, $\forall \alpha \in \mathbb{F}$

$\alpha f \in \mathcal{L}^p(X)$ because $(p<\infty)$ $\int_X |\alpha f(x)|^p \, d\mu = |\alpha|^p \underbrace{\int_X |f(x)|^p d\mu}_{f \in \mathcal{L}^p} < \infty$

$p = \infty$ $\quad$ $\text{ess sup} |\alpha f| = |\alpha| \, \text{ess sup} |f|$

$$|f(x) + g(x)|^p \leq \left(|f(x)| + |g(x)| \right)^p$$
$$\leq \left(2 \max \{ |f(x)|, |g(x)| \} \right)^p$$
$$= 2^p \left(|f(x)|^p + |g(x)|^p \right)$$

linearity of the integral

$$\int_X |f(x) + g(x)|^p \, d\mu \leq 2^p \left(\underbrace{\int_X |f(x)|^p d\mu}_{<\infty, \ f \in \mathcal{L}^p} + \underbrace{\int_X |g(x)|^p d\mu}_{<\infty, \ g \in \mathcal{L}^p} \right)$$

$< \infty$

$\Rightarrow f + g \in \mathcal{L}^p$ $\quad$ for $p < \infty$

for $p = \infty$

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \text{ess sup} |f| + \text{ess sup} |g|$$

a.e. $x \in X$

$\Rightarrow$ $\text{ess sup} |f+g| \leq \text{ess sup} |f| + \text{ess sup} |g|$.

It follows that $\mathcal{L}^p(X)$ is a vector space, $1 \leq p \leq \infty$.

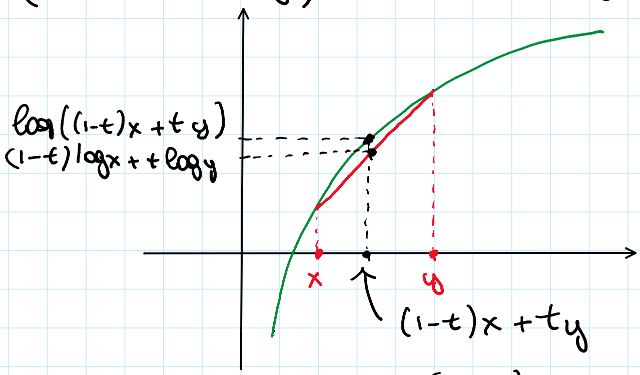
YOUNG'S INEQUALITY (for numbers)

Let $1 < p, q < \infty$, $\quad \frac{1}{p} + \frac{1}{q} = 1$ ("dual exponents or conjugate indices")

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \qquad \forall a, b \geq 0$$

Proof. $\log : (0, +\infty) \to \mathbb{R}$ is concave i.e.,

$$(\ast) \quad \log\big((1-t)x + ty\big) \geq (1-t)\log x + t \log y \qquad 0 \leq t \leq 1$$



take exp in $(\ast)$: $\quad (1-t)x + ty \geq x^{1-t} y^t$

Now choose $x = a^p \quad y = b^q \quad t = \frac{1}{q}$

$1 - t = 1 - \frac{1}{q} = \frac{1}{p}$

we get: $\quad \frac{1}{p}a^p + \frac{1}{q}b^q \geq (a^p)^{\frac{1}{p}}(b^q)^{\frac{1}{q}} = ab$

if $a, b > 0$

if either $a = 0$ or $b = 0 \implies ab = 0 \leq \frac{a^p}{p} + \frac{b^q}{q}$. ∎

HÖLDER'S INEQUALITY

Consider $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

If $f \in L^p(X)$, $g \in L^q(X)$ then $fg \in L^1(X)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \qquad \text{Hölder's inequality.}$$

Proof. 1) $\quad p = \infty \quad q = 1 \quad$ (similarly, $p = 1$ and $q = \infty$)

$$|f(x)g(x)| \leq |f(x)|\,|g(x)| \leq \big(\text{ess sup}|f|\big)\,|g(x)| \quad \text{a.e. } x$$

$$\|fg\|_1 = \int_X |f(x)g(x)|\,d\mu \leq \text{ess sup}|f| \int_X |g(x)|\,d\mu$$

$$= \|f\|_\infty \|g\|_1 < \infty$$

2) $1 < p, q < \infty$. Take $f \in L^p(X)$, $g \in L^q(X)$.

$\|f\|_p = 0 \implies f(x) = 0$ a.e. $x \implies |f(x)g(x)| = 0$ a.e. x

Hölder's ineq. is satisfied: $0 = 0$

similarly if $\|g\|_q = 0$ Hölder's ineq. is satisfied

Now assume $\|f\|_p, \|g\|_q > 0$ and use Young's inequality $a = \frac{|f(x)|}{\|f\|_p} \geq 0$, $b = \frac{|g(x)|}{\|g\|_q}$

$$\frac{|f(x)|}{\|f\|_p}\frac{|g(x)|}{\|g\|_q} \leq \frac{1}{p}\frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q}\frac{|g(x)|^q}{\|g\|_q^q}$$

$$\implies \frac{1}{\|f\|_p\|g\|_q}\int_X |f(x)g(x)|\,d\mu \leq \frac{1}{p\|f\|_p^p}\underbrace{\int_X |f(x)|^p d\mu}_{=\|f\|_p^p} + \frac{1}{q\|g\|_q^q}\underbrace{\int |g(x)|^q d\mu}_{=\|g\|_q^q}$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

So $\|fg\|_1 \leq \|f\|_p \|g\|_q$. ∎