Lemma. $(X, \|\cdot\|)$ normed space, Y finite-dimensional subspace of X. Then Y is closed.

Proof. $(Y, \|\cdot\|)$ normed space finite-dimensional $\Rightarrow$ Y is complete. Then Y is closed. (Recall Y complete $\Rightarrow$ Y closed.)

Theorem (Bolzano-Weierstrass) $(X, \|\cdot\|)$ finite-dimensional normed space, $K \subseteq X$ subset. Then

K compact $\Leftrightarrow$ K is closed and bounded.

Theorem. $(X, \|\cdot\|)$ infinite-dimensional normed space. Then neither $D = \{x \in X : \|x\| \leq 1\}$ nor $C = \{x \in X : \|x\| = 1\}$ is compact.

Remark. $F(x) := \|x\| : X \to \mathbb{R}$ continuous

$\Rightarrow$ $D = F^{-1}([0,1])$, $C = F^{-1}(\{1\})$

closed in $\mathbb{R}$

$\Rightarrow$ D and C are closed since F is continuous.
Clearly D and C are bounded.

Proof. We construct a sequence $\{x_n\} \subset C \; (\subset D)$ that does not have any convergent subsequence.

- Choose $x_1 \in X$: $\|x_1\| = 1$
- $X_1 := \operatorname{Span}\{x_1\}$ $\dim X_1 = 1$, X_1 subspace and $X_1 \neq X$ since X is infinite dimensional. Since X_1 is finite dimensional $\Rightarrow$ X_1 is closed by the previous lemma. Hence we can apply Riesz' lemma with $\alpha = \frac{1}{2}$ $\Rightarrow$ $\exists\, x_2 \in X$ such that $\|x_2\| = 1$ and $\|x_2 - y\| > \frac{1}{2}$, $\forall y \in X_1$. In particular, $\|x_2 - x_1\| > \frac{1}{2}$.

- $X_2 = \text{Span}\{x_1, x_2\}$ $\dim X_2 = 2$, X_2 is a subspace of X, $X_2 \neq X$, and X_2 is closed, by applying Riesz' lemma again ($\alpha = \frac{1}{2}$) $\exists\, x_3 \in X$: $\|x_3\| = 1$ and $\|x_3 - y\| > \frac{1}{2}$, $\forall y \in X_2$
  In particular, $\|x_3 - x_2\| > \frac{1}{2}$, $\|x_3 - x_1\| > \frac{1}{2}$.
- Iterating this argument we construct a sequence $\{x_n\} \subset X$: $\|x_n\| = 1$, $\forall n$ and $\|x_n - x_m\| > \frac{1}{2}$, $\forall n \neq m$.
- Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$ then $\|x_{n_k} - x_{n_\ell}\| > \frac{1}{2}$, $\forall k \neq \ell$
  $\Rightarrow \{x_{n_k}\}$ is not a Cauchy sequence, hence it is not convergent.
  (Recall: any convergent sequence is a Cauchy seq.). ∎

SERIES IN NORMED SPACES

$(X, \|\cdot\|)$ normed space. $\{x_n\} \subset X$.

$\sum_{n=1}^{\infty} x_n$ denotes the limit of the sequence of **partial sums** $S_N = \sum_{n=1}^{N} x_n = x_1 + x_2 + \cdots + x_N \in X$, $\forall N$

So that we can consider the sequence $\{S_N\} \subset X$

Def. If $S_N \to x \in X$, as $N \to +\infty$ we say that the series is convergent to x

Write: $\sum_{n=1}^{\infty} x_n = x$ **SUM of the series.**

Def. $\sum_{n=1}^{\infty} x_n$ is called **absolutely convergent** if $\sum_{n=1}^{\infty} \|x_n\|$ is convergent in $\mathbb{R}$.

Theorem. $(X, \|\cdot\|)$ Banach space. Then if $\sum_{n=1}^{\infty} x_n$ is absolutely convergent then $\sum x_n$ is convergent in X.

Proof. Let us show that $\{S_N\}_N$ is a Cauchy sequence:

$$\|S_N - S_M\| = \Big\|\sum_{n=M+1}^{N} x_n\Big\| \leq \sum_{n=M+1}^{N} \|x_n\| \qquad (*)$$

$M < N$

$\tilde{S}_N := \sum_{n=1}^{N} \|x_n\|$ we know by assumption that $\{\tilde{S}_N\}$ is convergent $\Rightarrow$ $\{\tilde{S}_N\}$ is a Cauchy seq.

$\forall \varepsilon > 0 \quad \exists N_0 : \quad \forall N > M \geq N_0 \quad |\tilde{S}_N - \tilde{S}_M| < \varepsilon$

$$|\tilde{S}_N - \tilde{S}_M| = \tilde{S}_N - \tilde{S}_M = \sum_{n=M+1}^{N} \|x_n\| < \varepsilon$$

Hence, by $(*)$ we have $\|S_N - S_M\| < \varepsilon$

Since X is a Banach space $\{S_N\}$ is convergent.

Theorem. $(X, \|\cdot\|)$ normed space. Then

X is a Banach space $\Leftrightarrow$ any absolutely convergent series is convergent in X.

Def. $\sum_n x_n$ is called unconditionally convergent to x if $\sum_n x_{\sigma(n)} = x$ for every permutation σ, i.e., $\sigma : \mathbb{N} \to \mathbb{N}$ bijection.

Let $(X, \|\cdot\|)$ be a Banach space. Then

$$\sum_{n=1}^{\infty} x_n \text{ absolutely conv.} \underset{\Longleftarrow}{\overset{\dim X < \infty}{}} \!\!\Longrightarrow \sum_{n=1}^{\infty} x_n \text{ unconditionally conv.}$$

$\sigma = id, \ \sigma(n) = n$

$$\sum_{n=1}^{\infty} x_n \text{ convergent}$$

In general, if $\sum_{n=1}^{\infty} x_n$ is conv. $\not\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ is absolutely conv.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

INNER PRODUCT SPACES

Def. X vector space over $\mathbb{R}$. $(\cdot,\cdot) : X \times X \to \mathbb{R}$ is called an *inner product* if

1) $(x,x) \geq 0, \quad \forall x$ (positive definiteness)
$(x,x) = 0 \iff x = 0$

2) $(x,y) = (y,x), \quad \forall x,y$ (symmetry)

3) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z)$ (linearity)
$\forall \alpha, \beta \in \mathbb{R}, \quad \forall x,y,z \in X$

NOTE: 2) + 3) $\Rightarrow$

$$(z, \alpha x + \beta y) \overset{2)}{=} (\alpha x + \beta y, z) \overset{3)}{=} \alpha(x,z) + \beta(y,z) \overset{2)}{=} \alpha(z,x) + \beta(z,y)$$

Example. $X = \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$

$(x,y) := \sum_{j=1}^{n} x_j y_j$ scalar product on $\mathbb{R}^n$

Def. X vector space over $\mathbb{C}$. $(\cdot,\cdot) : X \times X \to \mathbb{C}$ is called *inner product* if

1), 3) as above

2) $(x,y) = \overline{(y,x)}$ $\leftarrow$ complex conjugate (Hermitian symmetry)

Note: 2) + 3) $\Rightarrow$

$$(z, \alpha x + \beta y) \overset{2)}{=} \overline{(\alpha x + \beta y, z)} \overset{3)}{=} \overline{\alpha(x,z) + \beta(y,z)}$$
$$= \bar{\alpha}\,\overline{(x,z)} + \bar{\beta}\,\overline{(y,z)} \overset{2)}{=} \bar{\alpha}(z,x) + \bar{\beta}(z,y)$$

Example. $X = \mathbb{C}^n$ $\quad x = (x_1, \dots, x_n)$, $\quad y = (y_1, \dots, y_n)$

$$(x, y) = \sum_{j=1}^{n} x_j \overline{y_j}$$ check that it is an inner product

Def. X vector space with an inner product is called **inner product space**.

Example. X finite-dimensional vector space, $\dim X = n$.
Let $\{e_1, \dots, e_n\}$ be a basis for X.
$\forall x, y \in X$, $\quad x = \sum_{j=1}^{n} x_j e_j \quad y = \sum_{j=1}^{n} y_j e_j$

$$(x, y) = \sum_{j=1}^{n} x_j \overline{y_j}$$ it is an inner product on X.

Example. $X = \ell^2(\mathbb{N})$, For $(x_n)_n, (y_n) \in \ell^2(\mathbb{N})$
define the inner product: $\left((x_n), (y_n)\right) = \sum_{n=1}^{\infty} x_n \overline{y_n}$

First, we show that $(\cdot, \cdot)$ is well defined
The series is absolutely conv $\Rightarrow$ it is convergent
Hölder's inequ. $(p = q = 2)$

$$\sum_{n=1}^{\infty} |x_n \overline{y_n}| \le \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |y_n|^2\right)^{1/2} = \|(x_n)\|_{\ell^2} \|(y_n)\|_{\ell^2} < \infty$$

Moreover: $\quad x = (x_n) \in \ell^2$

1) $$(x, x) = \sum_{n=1}^{\infty} x_n \overline{x_n} = \sum_{n=1}^{\infty} |x_n|^2 = \|(x_n)\|_{\ell^2}^2 \ge 0$$

$$\Rightarrow (x, x) \ge 0 \quad \text{and} \quad (x, x) = 0 \Leftrightarrow \|x\|_{\ell^2}^2 = 0$$

$$\Leftrightarrow \|x\|_{\ell^2} = 0 \Leftrightarrow x = 0 \quad (x_n = 0, \forall n)$$

2) $$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n} = \sum_{n=1}^{\infty} \overline{\overline{x_n \overline{y_n}}} = \sum_{n=1}^{\infty} \overline{\overline{x_n} y_n}$$

$$= \overline{\sum_{n=1}^{\infty} \overline{x_n} y_n} = \overline{(y, x)}$$

$$z \mapsto \overline{z} : \mathbb{C} \to \mathbb{C} \text{ is continuous}$$

3) $$(\alpha x+\beta y, z) = \sum_{n=1}^{\infty} (\alpha x_n+\beta y_n)\overline{z}_n = \sum_{n=1}^{\infty}(\alpha x_n\overline{z}_n + \beta y_n\overline{z}_n)$$

absolutely convergent

$$= \sum_{n=1}^{\infty} \alpha x_n \overline{z}_n + \sum_{n=1}^{\infty} \beta y_n \overline{z}_n$$

$$= \alpha \sum_{n=1}^{\infty} x_n\overline{z}_n + \beta \sum_{n=1}^{\infty} y_n \overline{z}_n$$

$$= \alpha(x,z) + \beta(y,z)$$

Example. $L^2(X)$, (X, Σ, μ) measure space

inner product: $\forall f, g \in L^2(X)$, $(f,g) = \int_X f\overline{g}\,d\mu$

$$\int_X |f\overline{g}|\,d\mu \overset{\text{Hölder's ineq.}}{\leq} \left(\int_X |f|^2 d\mu\right)^{1/2} \left(\int_X |g|^2 d\mu\right)^{1/2} < \infty$$

check as exercise that it is an inner product.

Example. $(X, (\cdot,\cdot)_X)$ inner product space, $Y \subseteq X$ subspace. Then

$$(x,y)_Y := (x,y)_X, \quad \forall x,y \in Y$$

defines an inner product on Y, so-called **induced inner product**.

Theorem. $(X, (\cdot,\cdot))$ inner product space. Then:

1) CAUCHY-SCHWARZ INEQUALITY:

(CS) $|(x,y)|^2 \leq (x,x)\cdot(y,y)$, $\forall x,y \in X$

2) $\|x\| := \sqrt{(x,x)}$ defines a norm on X, so-called **induced norm from the inner product**.

Proof. 1) Step 1. x, y linearly dipendent, say $x = \lambda y$

$$|(x,y)|^2 = (x,y)\overline{(x,y)} = (\lambda y, y)\overline{(\lambda y, y)} = \lambda(y,y)\overline{\lambda}\overline{(y,y)}$$

$$= \lambda\overline{\lambda}(y,y)(y,y) = (\underbrace{\lambda y}_{=x}, \underbrace{\lambda y}_{=x})(y,y) = (x,x)(y,y)$$

(CS) is satisfied with "="

Step 2. x, y lin. independent $\Rightarrow x \neq \lambda y$, $\forall \lambda$

$$0 \overset{2)}{<} (x-\lambda y, x-\lambda y) = (x,x) - \lambda(y,x) - \overline{\lambda}(x,y) + \lambda\overline{\lambda}(y,y)$$

Choose $\lambda = \dfrac{(x,y)}{(y,y)}$ note $(y,y) \neq 0$ since $y \neq 0$

So

$$0 < (x,x) - \frac{|(x,y)|^2}{(y,y)} - \cancel{\frac{|(x,y)|^2}{(y,y)}} + \cancel{\frac{|(x,y)|^2}{(y,y)}}$$

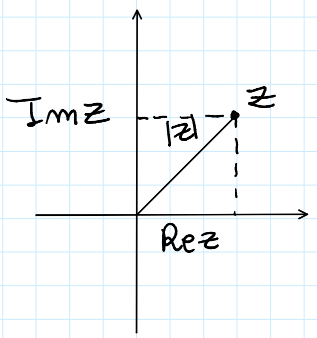
$$|(x,y)|^2 < (x,x)(y,y)$$

strict inequality!

2) $\|x\| = \sqrt{(x,x)}$ is a norm:

- $\|x\| \geq 0$ and $\|x\| = 0 \iff \sqrt{(x,x)} = 0 \iff (x,x) = 0$
  $\iff x = 0$
- $\|\alpha x\|^2 = (\alpha x, \alpha x) = \alpha\bar{\alpha}(x,x) = |\alpha|^2 \|x\|^2$
  $\Rightarrow \|\alpha x\| = |\alpha| \, \|x\|$, $\forall \alpha \in \mathbb{C}$, $\forall x$
- $\|x+y\|^2 = (x+y, x+y) = (x,x) + \underbrace{(x,y) + \underbrace{(y,x)}_{=\overline{(x,y)}}}_{2\,\mathrm{Re}(x,y)} + (y,y)$

Recall: $|\mathrm{Re}\, z| \leq |z|$



$\mathbb{C}$

$|\mathrm{Re}\, z| \leq |z|$

$$\begin{aligned}
\|x+y\|^2 &\leq (x,x) + 2|(x,y)| + (y,y) \\
&\underset{(CS)}{\leq} (x,x) + 2\sqrt{(x,x)}\sqrt{(y,y)} + (y,y) \\
&= \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$, $\forall x, y$

$\Rightarrow$ Every inner product space $(X, (\cdot,\cdot)_X)$ is a normed space with $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.

The converse is not true in general!