Lecture 9

$A \subseteq X$, X inner product space

$$A^\perp = \{ x \in X : (x,a)=0, \forall a \in A \}$$

Principal properties of $A^\perp$

1) $0 \in A^\perp$
2) If $0 \in A$ then $A \cap A^\perp = \{0\}$, otherwise $A \cap A^\perp = \emptyset$
3) $\{0\}^\perp = X$, $X^\perp = \{0\}$
4) If $\exists a \in X, r>0 \mid B_a(r) \subset A$, then $A^\perp = \{0\}$; in particular, if A is a non-empty open set then $A^\perp = \{0\}$.
5) If $B \subseteq A$ then $A^\perp \subseteq B^\perp$.
6) $A^\perp$ is a closed subspace of X.
7) $A \subseteq (A^\perp)^\perp$

Proof. 1) $0 \in A^\perp$: $(0,a)=0$, $\forall a \in A$

2) By contradiction, assume $\exists x \neq 0 \mid x \in A \cap A^\perp$

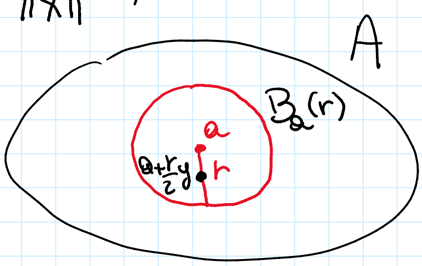
$$(\underset{\substack{\Cap \\ A}}{x}, \underset{\substack{\Cap \\ A^\perp}}{x}) = 0 \iff x = 0$$

3) Trivial

4) By contradiction assume $\exists x \in A^\perp$, $x \neq 0$.

Define $y = \frac{x}{\|x\|}$, then $\|y\| = 1$ $(y,a) = \frac{1}{\|x\|}(x,a) = 0$ $\forall a \in A$

$y \in A^\perp$



$a + \frac{r}{2} y \in B_a(r)$

$$\| \cancel{a} + \frac{r}{2} y - \cancel{a} \| = \frac{r}{2} \underbrace{\|y\|}_{=1} = \frac{r}{2} < r \Rightarrow a + \frac{r}{2} y \in B_a(r) \subset A$$

$$\underbrace{(\underset{\substack{\cap \\ A^{\perp}}}{y}, \underbrace{a+\tfrac{r}{2}y}_{\in A})}_{=0} = \underbrace{(\underset{\substack{\cap \\ A^{\perp}}}{y}, \underset{\substack{\cap \\ A}}{a})}_{=0} + \frac{r}{2}\underbrace{(y,y)}_{\|y\|^2 = 1} \Rightarrow \frac{r}{2} = 0$$

contradiction!

5) $x \in A^{\perp} \Rightarrow (x,b) = 0 \quad \forall b \in B \subseteq A$

$\Rightarrow x \in B^{\perp}$

6) $A^{\perp}$ is a subspace of X: $\forall \alpha, \beta \in \mathbb{F}$, $\forall x, y \in A^{\perp}$

$$(\alpha x + \beta y, a) \overset{\text{lin.}}{=} \alpha \underbrace{(x,a)}_{=0} + \beta \underbrace{(y,a)}_{=0} = 0, \quad \forall a \in A$$

hence $\alpha x + \beta y \in A^{\perp}$

$A^{\perp}$ is closed: consider a sequence $\{x_n\} \subset A^{\perp}$ such that $x_n \to x \in X$, for $n \to \infty$ then we have to show that $x \in A^{\perp}$. $\forall a \in A$,

$$(x,a) = (\lim_{n\to\infty} x_n, a) \overset{\text{continuity of } (\cdot,\cdot)}{=} \lim_{n\to\infty} \underbrace{(x_n, a)}_{=0,\ \forall n} = 0$$

$\Rightarrow x \in A^{\perp}$.

7) $a \in A$, $\forall x \in A^{\perp}$ $(\underset{\substack{\cap \\ A}}{a}, \underset{\substack{\cap \\ A^{\perp}}}{x}) = 0 \Rightarrow a \in (A^{\perp})^{\perp}$.

Lemma (Characterization of the orthogonal complement for linear subspaces)

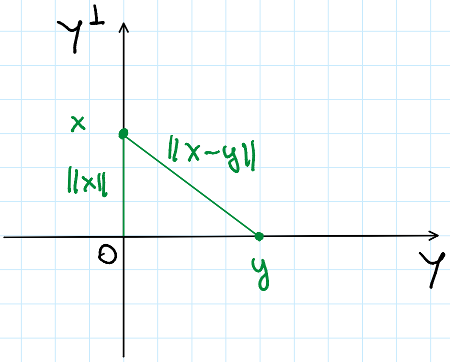
Consider $Y \subseteq X$ subspace of an inner product space X.

Then $x \in Y^{\perp} \iff \|x - y\| \geq \|x\|, \quad \forall y \in Y$

Ex. $X = \mathbb{R}^2$

$Y = Sp\{e_1\}$

$Y^{\perp} = Sp\{e_2\}$



Proof. $\Rightarrow$ $x \in Y^{\perp}$, $\forall y \in Y$, $(x, y) = 0$

$$\|x-y\|^2 = (x-y, x-y) = \|x\|^2 - \underbrace{(x, y)}_{=0} - \underbrace{(y, x)}_{=0} + \|y\|^2$$

$$= \|x\|^2 + \underbrace{\|y\|^2}_{\geq 0} \geq \|x\|^2$$

then $\|x-y\| \geq \|x\|$.

$\Leftarrow$ Assume $\|x\|^2 \leq \|x-y\|^2$, $\forall y \in Y$

Y is a subspace, $\forall \alpha \in \mathbb{F}$, $\forall y \in Y \Rightarrow \alpha y \in Y$

so $\|x\|^2 \leq \|x - \alpha y\|^2$, in particular it holds for $\alpha \in \mathbb{R}$, in this case:

$$\cancel{\|x\|^2} \leq (x - \alpha y, x - \alpha y) = \cancel{\|x\|^2} - \alpha(x, y) - \alpha \overbrace{(y, x)}^{=\overline{(x,y)}} + \alpha^2 \|y\|^2$$

$$\alpha^2 \|y\|^2 - 2\alpha \,\mathrm{Re}(x, y) \geq 0 \qquad (*)$$

If $\alpha > 0$ $\quad \mathrm{Re}(x, y) \leq \frac{\alpha}{2} \|y\|^2$, $\forall \alpha > 0$

letting $\alpha \to 0$ $\Rightarrow$ $\mathrm{Re}(x, y) \leq 0$ (1)

If $\alpha < 0$ $\quad \mathrm{Re}(x, y) \geq \frac{\alpha}{2} \|y\|^2$, $\forall \alpha < 0$

letting $\alpha \to 0$ $\Rightarrow$ $\mathrm{Re}(x, y) \geq 0$ (2)

So from (1) and (2) it follows that

$\mathrm{Re}(x, y) = 0$ (3)

If X is a real inner product space:

$\mathrm{Re}(x, y) = (x, y) = 0 \Rightarrow x \in Y^{\perp}$

If X is complex, $(x, y) = \rho e^{i\theta}$ for a certain $\rho > 0$, $\theta \in \mathbb{R}$. $e^{-i\theta}(x, y) = \rho \in \mathbb{R}$

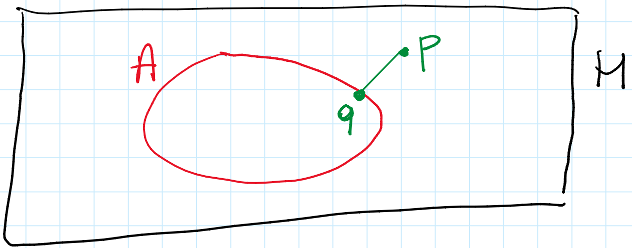
$\Leftrightarrow (x, e^{i\theta} y) \in \mathbb{R}$, $y \in Y \Rightarrow e^{i\theta} y \in Y$ because Y is a subspace, so by (3)

$$(x, e^{i\theta} y) = 0 \Leftrightarrow \underbrace{e^{-i\theta}}_{\neq 0}(x, y) = 0$$

$$\iff (x,y) = 0 \ . \ \blacksquare$$

Theorem. (Projection Theorem).
H Hilbert space. Let $A \subset H$ be a non-empty closed convex subset of H. For every $p \in H$ $\exists!\, q \in A$:

$$\|p-q\| = \underbrace{\inf\{\|p-a\|,\ a\in A\}}_{:= dist(p,A)} \qquad (PT)$$



Proof. We first prove the existence of q.
Let us call $\gamma := \inf\{\|p-a\|,\ a \in A\}$
↑ it is not empty since $A \neq \emptyset$

the infimum γ is well defined ad $\gamma \geq 0$
By def. of infimum, $\forall\, n \in \mathbb{N}_+$ $\exists\, q_n \in A$:

$$\gamma^2 \leq \|p-q_n\|^2 < \gamma^2 + \frac{1}{n} \qquad (*)$$

So we have constructed a sequence $\{q_n\} \subset A$
$\{q_n\}$ is a Cauchy sequence: use the pallalegram rule with the elements $p-q_n$ and $p-q_m$:

$$\|(p-q_n)+(p-q_m)\|^2 + \|(\cancel{p}-q_n)-(\cancel{p}-q_m)\|^2 = 2\|p-q_n\|^2 + 2\|p-q_m\|^2$$

$$\|2p-(q_n+q_m)\|^2 + \|q_n-q_m\|^2 < 2\gamma^2 + \frac{2}{n} + 2\gamma^2 + \frac{2}{m}$$

$$4\left\|p - \frac{q_n+q_m}{2}\right\|^2 + \|q_n-q_m\|^2 < 4\gamma^2 + 2\left(\frac{1}{n}+\frac{1}{m}\right) \qquad (**)$$

A is convex $\frac{q_n+q_m}{2} \in A$ and $\left\|p - \frac{q_n+q_m}{2}\right\|^2 \geq \gamma^2$

$$\Rightarrow \quad -4\left\|p - \frac{q_n+q_m}{2}\right\|^2 \leq -4\gamma^2$$

so $(**)$ becomes :

$$\|q_n - q_m\|^2 < \cancel{4\gamma^2} - \cancel{4\gamma^2} + 2\left(\frac{1}{n}+\frac{1}{m}\right)$$

$$\Leftrightarrow \quad \|q_n - q_m\|^2 < 2\left(\frac{1}{n}+\frac{1}{m}\right), \quad \forall n, m \in \mathbb{N}_+$$

$\{q_n\}$ is a Cauchy seq. since $\forall \varepsilon > 0$ you can find $N_0 \in \mathbb{N}_+ \mid$ $\forall n, m \geq N_0$ $\quad 2\left(\frac{1}{n}+\frac{1}{m}\right) < \varepsilon^2$ hence $\|q_n - q_m\| < \varepsilon$

$\{q_n\} \subset A \subset H$ $\quad$ H is a Hilbert space : every Cauchy sequence is convergent in H : $\exists q \in H$ such that $q = \lim_{n\to\infty} q_n$. Moreover $q \in A$ since A is closed.

Using the inequalities in $(*)$

$$\lim_{n\to\infty} \gamma^2 + \frac{1}{n} = \gamma^2$$

$$\lim_{n\to\infty} \|p - q_n\|^2 \overset{\|\cdot\|\text{ cont.}}{=} \|p - \lim_{n\to\infty} q_n\|^2 = \|p-q\|^2$$

By the comparison theorem in $(*)$ we have $\|p-q\|^2 = \gamma^2$, i.e. $\|p-q\| = \gamma$

Secondly, let us show that q is unique. Assume $\exists w \in A : \|p-w\| = \gamma$. Use the parallelogram rule for $p-q$ and $p-w$:

$$\|(p-q)+(p-w)\|^2 + \|(\cancel{p}-q)-(\cancel{p}-w)\|^2 = 2\overbrace{\|p-q\|^2}^{=\gamma^2} + 2\overbrace{\|p-w\|^2}^{=\gamma^2}$$

$$4\left\|p - \frac{q+w}{2}\right\|^2 + \|q-w\|^2 = 4\gamma^2$$

$\frac{q+w}{2} \in A$ $\qquad 4\left\|p - \frac{q+w}{2}\right\| \geq 4\gamma^2$

$$\|q-w\|^2 = 4\gamma^2 - 4\left\|p - \frac{q+w}{2}\right\|^2 \leq 4\gamma^2 - 4\gamma^2 = 0$$

$$\Rightarrow \quad \|q-w\| = 0 \quad \Leftrightarrow \quad q = w.$$

Theorem. (The Orthogonal Decomposition Theorem)

H Hilbert space, $Y \subset H$ closed subspace of H. For every $x \in H$ $\exists!\,(y,z)$, $y \in Y$, $z \in Y^{\perp}$ with $x = y+z$. Also $\|x\|^2 = \|y\|^2 + \|z\|^2$.

Proof. Y is a subspace $\Rightarrow$ Y is not empty and convex. Given $x \in X$ $\exists!\, y \in Y$: $\|x-y\| \leq \|x-u\|$, $\forall u \in Y\ (\cdot)$ by the Projection Theorem. Define $z := x-y$ hence $x = y+z$. Let us show that $z \in Y^{\perp}$.

$$\forall u \in Y,\ \|z-u\| = \|x-y-u\| = \|x-(\underbrace{y+u}_{\in Y})\| \overset{(\cdot)}{\geq} \|x-y\| = \|z\|$$

so $\|z-u\| \geq \|z\|$, $\forall u \in Y$

$\Longleftrightarrow z \in Y^{\perp}$ by the characterization of the orthogonal complement for linear subspaces.

Let us show the uniqueness: assume

$$x = y_1 + z_1 = y_2 + z_2, \qquad y_1, y_2 \in Y,\ z_1, z_2 \in Y^{\perp}$$

then $\underbrace{y_1 - y_2}_{\in Y} = \underbrace{z_2 - z_1}_{\in Y^{\perp}}$, $Y, Y^{\perp}$ subspaces

recall $Y \cap Y^{\perp} = \{0\}$ hence $y_1 = y_2$ and $z_2 = z_1$.

$$\|x\|^2 = \|y+z\|^2 = (y+z, y+z) = \|y\|^2 + \underbrace{(z,y)}_{=0} + \underbrace{(y,z)}_{=0} + \|z\|^2$$

hence $\|x\|^2 = \|y\|^2 + \|z\|^2$. ■