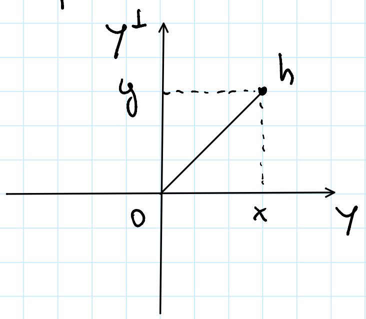
Orthogonal Decomposition Theorem

H Hilbert space, $Y \subset H$ closed subspace of H. Then $\forall x \in H$ $\exists!\ (y,z)$, $y \in Y$, $z \in Y^\perp$ and $x = y+z$. Moreover $\|x\|^2 = \|y\|^2 + \|z\|^2$.

Notation. We write $H = Y \oplus Y^\perp$
$\uparrow$ direct sum

The decomposition $x = y+z$ is called "the orthogonal decomposition of x w.r.to the subspace Y.

Remark. $H = \mathbb{R}^2$ $Y = \{(x,0),\ x \in \mathbb{R}\}$
$Y^\perp = \{(0,y),\ y \in \mathbb{R}\}$ $\forall h \in \mathbb{R}^2$



$|h|^2 = |x|^2 + |y|^2$
$h^2 = x^2 + y^2$.

Corollary. If H is a Hilbert space and $Y \subset H$ is a closed subspace of H then

$$Y^{\perp\perp} := (Y^\perp)^\perp = Y \qquad (*).$$

Corollary. H Hilbert space, $Y \subset H$ subspace of H. Then $Y^{\perp\perp} = \overline{Y}$.

Remark: if Y is closed, we come back to $(*)$.

Homework: Ex. 1-3-4.

Exercise. $X = \mathbb{R}^k$, $A = \{a\}$, with $a \in \mathbb{R}^k \setminus \{0\}$. Compute $A^\perp$.

Solution. Since $a \in \mathbb{R}^k$, $a = (a_1, \dots, a_k)$, $a_j \in \mathbb{R}$, $\forall j = 1, \dots, k$.

$$A^\perp = \{x = (x_1, \dots, x_k) \in \mathbb{R}^k : (x,a) = 0\}$$
$$= \left\{\sum_{j=1}^{k} a_j x_j = 0\right\}$$

Exercise. $X = \ell^2(\mathbb{N})$, $A = \{ y = (y_n) \in \ell^2 \mid y_{2n} = 0, \forall n \}$

Show that $A^\perp = \{ x = (x_n) \in \ell^2 \mid x_{2n+1} = 0, \forall n \}$

Solution. $x \in A^\perp \iff (x, y) = 0, \quad \forall y \in A$

$$\iff \sum_{n=1}^{\infty} x_n \overline{y_n} = 0$$

$$\iff \sum_{n=1}^{\infty} x_{2n+1} \overline{y_{2n+1}} = 0 \qquad (L)$$

$$S := \{ x = (x_n) \in \ell^2 \mid x_{2n+1} = 0, \forall n \}$$

We want to show: $S = A^\perp$.

First, $S \subseteq A^\perp$; in fact $x = (x_n) \in S$ then $x_{2n+1} = 0$, $\forall n$, then condition (L) is satisfied.

Now, we show $A^\perp \subseteq S$. By contradiction, assume $\exists\, x \in A^\perp \mid x \notin S \Rightarrow \exists\, m \in \mathbb{N} \mid x_{2m+1} \neq 0$. Then consider the sequence $e_{2m+1} = (0, \dots, 0, \underset{\substack{\uparrow \\ 2m+1}}{1}, 0, \dots, 0)$

then $e_{2m+1} \in A$ and we have:

$(\underset{\substack{\cap \\ A^\perp}}{x}, \underset{\substack{\cap \\ A}}{e_{2m+1}}) = 0$ But $(x, e_{2m+1}) = x_{2m+1} \neq 0$,

that is a contradiction. Hence $A^\perp \subseteq S$ and since $S \subseteq A^\perp \Rightarrow A^\perp = S$.

Exercise. X inner product space with inner product $(\cdot, \cdot)$. $A \subset X$ a subset of X. Show:

$$A^\perp = \overline{A}^\perp \qquad (**)$$

Solution. Observe that $A \subseteq \overline{A} \Rightarrow \overline{A}^\perp \subseteq A^\perp$

We have to show $A^\perp \subseteq \overline{A}^\perp$

Consider $x \in A^\perp \Rightarrow \forall y \in A \quad (x, y) = 0$

$y \in \overline{A}^{\perp}$, then there exists a sequence $\{y_n\} \subset A$ such that $y_n \to A$

$$(x, y) = (x, \lim_{n\to\infty} y_n) \overset{\text{by the continuity of } (\cdot,\cdot)}{=} \lim_{n\to\infty} \underbrace{(\underset{A^{\perp}}{x}, \underset{A}{y_n})}_{=0} = 0$$

$\Rightarrow x \in \overline{A}^{\perp}$.

Exercise. H Hilbert space. Y closed subspace of H, $Y \neq H$. Show $Y^{\perp} \neq \{0\}$.

Is this always true if Y is not closed? (Hint: Y dense, non-closed subspace of H).

Solution. $\{0\}^{\perp} = H$

We use $Y^{\perp\perp} = Y$ by $(*)$

If by contradiction, $Y^{\perp} = \{0\}$ then $Y = Y^{\perp\perp} = \{0\}^{\perp} = H$ then $Y = H$ that is a contradiction.

If Y is dense in $H \Rightarrow \overline{Y} = H$ and since by $(**)$ $Y^{\perp} = \overline{Y}^{\perp} = H^{\perp} = \{0\}$

As example $H = \ell^2$, $Y = c_{00}$

c_{00} is not closed in ℓ^2. Consider $x = \left(\frac{1}{n}\right)_{n\in\mathbb{N}_+} \in \ell^2$ because $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, of course $x \notin c_{00}$

$$x^1 = (1, 0, 0, \dots, 0, \dots) \in c_{00}$$

$$x^2 = (1, \tfrac{1}{2}, 0, \dots, 0, \dots) \in c_{00}$$

$$\vdots$$

$$x^n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, 0, 0, \dots) \in c_{00}$$

$\{x^n\} \subset c_{00}$ $\quad x^n \to x$ in ℓ^2:

$$\|x^n - x\|^2_{\ell^2} = \sum_{j=n+1}^{\infty} \frac{1}{j^2} \to 0, \quad n \to \infty$$

hence c_{00} is not closed.

c_{00} is dense in ℓ^2. $\forall x = (x_n) \in \ell^2 \Rightarrow \sum_{n=1}^{\infty} |x_n|^2 < \infty$. Then $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ /

$$\sum_{n \geq N} |x_n|^2 < \varepsilon^2 .$$ Then define $y \in c_{00}$

such that $y = (x_1, x_2, \dots, x_N, 0, 0, \dots, 0, \dots) \in c_{00}$

$$\|x - y\|_{\ell^2}^2 = \sum_{n \geq N} |x_n|^2 < \varepsilon^2 .$$

Exercise (MMWZ, Ex. 4)

H Hilbert space, $A \subset H$ non-empty set. Then

1) $A^{\perp\perp} = \overline{Sp A}$

2) $A^{\perp\perp\perp} = A^{\perp}$

Solution. 1) $A \subseteq \overline{Sp(A)} \Rightarrow \overline{Sp(A)}^{\perp} \subseteq A^{\perp}$

$$A^{\perp\perp} \subseteq \overline{Sp(A)}^{\perp\perp} = \overline{Sp(A)}$$

$\uparrow$ closed subspace $\Rightarrow$ use $Y^{\perp\perp} = Y$

So $A^{\perp\perp} \subseteq \overline{Sp(A)}$, we have to show: $\overline{Sp(A)} \subseteq A^{\perp\perp}$

We know that $A^{\perp\perp}$ is a closed subspace which contains A. Recall: $\overline{Sp(A)}$ is the smallest closed subspace containing the set $A \Rightarrow \overline{Sp(A)} \subseteq A^{\perp\perp}$

$\Rightarrow \overline{Sp(A)} = A^{\perp\perp}$.

2) $A^{\perp}$ is a closed subspace $\Rightarrow (\underbrace{A^{\perp}}_{Y})^{\perp\perp} = \underbrace{A^{\perp}}_{Y}$

ORTHONORMAL BASES in ∞-DIMENSIONS

Def. $(X, (\cdot,\cdot))$ inner product space, $\{e_n\} \subset X$ is said to be an *orthonormal sequence* (o.n.s) if $\|e_n\| = 1$, $\forall n$ and $(e_n, e_m) = 0$ for $n \neq m$.

In other words, $(e_n, e_m) = \delta_{nm}$, $\forall n, m$.

Example. $X = \ell^2$ $\quad \delta_n := (\delta_{nm})_{m \in \mathbb{N}}$

$$= (0, 0, \dots, 0, \underset{n^{th} \text{ entry}}{1}, 0, 0, \dots)$$

$\{\delta_n\}_{n\in\mathbb{N}} \subset \ell^2$

$\delta_n \in \ell^2$, $\forall n$ since $\sum_{m=0}^{\infty} |\delta_{nm}|^2 = 1^2 = 1$

so $\|\delta_n\|_{\ell^2} = 1$, $\forall n$

$(\delta_n, \delta_m) \underset{n\neq m}{=} 0 \quad \Rightarrow \{\delta_n\}$ is an o.n.s. in ℓ^2.

Example $L^2_{\mathbb{C}}([-\pi,\pi]) = \{ f : \mathbb{R} \to \mathbb{C}$ measurable and 2π-periodic such that $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \}/\sim$

$f \sim g \iff f(x) = g(x)$ a.e. $x \in \mathbb{R}$

Consider $e_n(x) := \frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$

$\{e_n\}_{n\in\mathbb{Z}} \subset L^2([-\pi,\pi])$

$$\|e_n\|_2^2 = \int_{-\pi}^{\pi} |e_n(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 \, dx = \frac{1}{2\pi}\cdot 2\pi = 1,$$

$\forall n \in \mathbb{N}$

$$(e_n, e_m) \overset{n\neq m}{=} \int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x} dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{i(n-m)} \underbrace{e^{i(n-m)x}}_{\text{it is } 2\pi\text{-periodic!}} \right]_{-\pi}^{\pi}$$

$$= 0,$$

Observe that $\{e_n\}$ is an o.n.s. for any $L^2_{\mathbb{C}}([a,b])$ for $b-a = 2\pi$.

Properties of o.n.s.

1) $\{e_n\}$ o.n.s. $\Rightarrow \{e_n\}$ is linearly independent.

Proof. Take $\alpha_n \in \mathbb{F}$ / $\sum_{n=1}^{k} \alpha_n e_n = 0$

$m \in \{1,\dots,k\}$ $\quad 0 = \Big(\underbrace{\sum_{n=1}^{k} \alpha_n e_n}_{=0}, e_m \Big) = \sum_{n=1}^{k} \alpha_n \underbrace{(e_n, e_m)}_{=\delta_{nm}} = \alpha_m$

$\Rightarrow \alpha_m = 0 \quad , \quad \forall m = 1, \dots, k.$

Observe that $\{e_n\} \subset X$ $\quad \{e_n\}$ o.n.s.
then X must be infinite dimensional

Theorem. Any infinite-dimensional inner product space X contains an o.n.s. $\{e_n\}$.

Sketch of proof. Take $x_1 \in X : \|x_1\| = 1$.

- $X_1 = Sp\{x_1\}$ $\quad \dim X_1 = 1 \Rightarrow X_1 \neq X$
  X_1 is a closed (X_1 finite-dim. subspace) subspace.
- By Riesz' lemma, $\exists x_2 \in X$, $\|x_2\| = 1$ /
  $\|x_2 - y\| > \frac{1}{2}$, $\forall y \in Sp\{x_1\}$. In particular
  $\|x_2 - x_1\| > \frac{1}{2}$ $\quad x_2 \notin Sp\{x_1\}$
- $X_2 = Sp\{x_1, x_2\}$ then X_2 is a closed subspace of X, $X_2 \neq X \Rightarrow$ by Riesz' lemma $\exists x_3 \in X$
  $\|x_3\| = 1$ / $\|x_3 - y\| > \frac{1}{2}$, $\forall y \in X_2$
- Iterating this argument, we construct a sequence $\{x_n\}$ of linearly independent vectors / $\{x_n\} \subset X$
  By inductively applying the Gram-Schmidt algorithm you construct the sequence $\{e_n\}$.

Question: If $\{e_n\}$ is an o.n.s. for X, is it possible to write:

$$x = \sum_n (x, e_n) e_n \quad , \quad \forall x \in X$$

1) Does the series $\sum_n (x, e_n) e_n$ converge in X?
2) If it converges, does it converge to x?