Fourier Series

$L^2_{\mathbb{C}}([-\pi,\pi]) := \{ f : \mathbb{R} \to \mathbb{C},\ 2\pi\text{-periodic, measurable and such that } \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \} / \sim$

$f \sim g \iff f(x) = g(x) \quad a.e.\ x \in \mathbb{R}$

$\{e_n\}_{n \in \mathbb{Z}} \subset L^2_{\mathbb{C}}([-\pi,\pi])$, $\quad e_n(x) := \frac{1}{\sqrt{2\pi}} e^{inx}$

Definition. A trigonometric polynomial p is any element of the $\mathrm{Sp}\{e_n\}$, that is,

$$p(x) = \sum_{m=-M}^{N} \alpha_m e_m, \quad \alpha_m \in \mathbb{C}, \text{ with } M, N \in \mathbb{N}$$

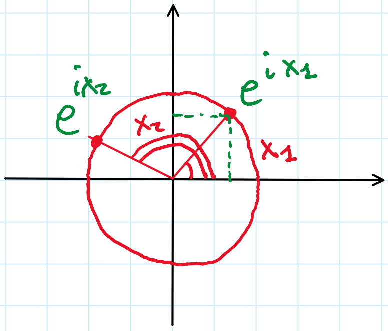
So $p(x)$ is a linear combination of powers of $\sin x$ and $\cos x$ (trigonometric functions).

$C(\mathbb{T}) := \{ f : \mathbb{R} \to \mathbb{C} \text{ continuous and } 2\pi\text{-periodic} \}$

Theorem. $\mathrm{Sp}\{e_n\} \subset C(\mathbb{T})$ and it is dense w.r.t. the uniform norm $\|\cdot\|_\infty$.

Proof. By Stone-Weierstrass Theorem, it is enough to show that $\mathrm{Sp}\{e_n\}$ is a unital algebra which separates points on $[-\pi,\pi)$, closed under conjugation.

- $\mathrm{Sp}\{e_n\}$ is a subspace of $C(\mathbb{T})$
- $\forall p, q \in \mathrm{Sp}\{e_n\}$, $pq \in \mathrm{Sp}\{e_n\}$ (algebra)
- $1 = e^{i0x} \in \mathrm{Sp}\{e_n\}$
- $\mathrm{Sp}\{e_n\}$ separates points on $[-\pi,\pi)$



$e^{ix_1} \neq e^{ix_2}$

- $p \in \mathrm{Sp}\{e_n\} \Rightarrow \overline{p} \in \mathrm{Sp}\{e_n\}$.

Hence, by Stone-Weierstrass theorem, $\overline{\mathrm{Sp}\{e_n\}}^{\|\,\|_\infty} =$
$= C(\mathbb{T})$.

The proof is concluded. ∎

Property. $C(\mathbb{T}) \subset L^2_{\mathbb{C}}([-\pi,\pi])$ is dense in $L^2_{\mathbb{C}}([-\pi,\pi])$ w.r.t. the L^2-norm.

Corollary. $\{e_n\}$ is complete in $L^2([-\pi,\pi])$.
Equivalently, $\{e_n\}$ is an o.n.b. for $L^2([-\pi,\pi])$.

Proof $\forall f \in L^2_{\mathbb{C}}([-\pi,\pi])$, $\forall \varepsilon > 0$ we will find an element $p \in \mathrm{Sp}\{e_n\} \mid \|f-p\|_2 < \varepsilon$.

By the density of $C(\mathbb{T})$ in $L^2_{\mathbb{C}}([-\pi,\pi])$, there exists $g \in C(\mathbb{T})$ such that $\|f-g\|_2 < \frac{\varepsilon}{2}$

$h \in C(\mathbb{T}) \subset L^2([-\pi,\pi])$

$$\|h\|_2^2 = \int_{-\pi}^{\pi} |h(x)|^2 dx \le \|h\|_\infty^2 \int_{-\pi}^{\pi} dx = 2\pi \|h\|_\infty^2$$

$\|h\|_2 \le \sqrt{2\pi}\, \|h\|_\infty$ $\quad (2-\infty)$

Given the $g \in C(\mathbb{T}) \mid \|f-g\|_2 < \frac{\varepsilon}{2}$, by the density of $\mathrm{Sp}\{e_n\}$ in $C(\mathbb{T})$, there exists a trigonometric polynomial $p \in \mathrm{Sp}\{e_n\} \mid$
$\|g-p\|_\infty < C\frac{\varepsilon}{2}$ with $C > 0$ to be determined

$$\|f-p\|_2 = \|f-g+g-p\|_2 \le \|f-g\|_2 + \|g-p\|_2$$

$$\underset{(2-\infty)}{\le} \|f-g\|_2 + \sqrt{2\pi}\,\|g-p\|_\infty$$

$$< \frac{\varepsilon}{2} + \sqrt{2\pi}\, c\, \frac{\varepsilon}{2} < \varepsilon$$

provided that we choose $c = \frac{1}{\sqrt{2\pi}}$. ∎

For $f \in L^2([-\pi,\pi])$ "the m^{th} - Fourier coefficient" of f: $\hat{f}(m) = (f, e_m) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-imx}\,dx$, $m \in \mathbb{Z}$.

Hence, since $\{e_n\}$ is an o.n.b., any f admits the representation:

$$f = \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e_m$$ Fourier series expansion of f

with unconditional convergence in $L^2_{\mathbb{C}}([-\pi,\pi])$

Parseval's Theorem: $\|f\|_2^2 = \sum_{m\in\mathbb{Z}} |\hat{f}(m)|^2$

$$= \|(\hat{f}(m))_{m\in\mathbb{Z}}\|^2_{\ell^2(\mathbb{Z})}$$

Corollary. The set of functions

$$E = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\sin mx \right\}_{m\in\mathbb{N}_+}$$

is an o.n.b. for $L^2_{\mathbb{C}}([-\pi,\pi])$.

Proof. Check that E is an o.n. set.

Using Euler formula $e^{imx} = \cos mx + i\sin mx$, $m \in \mathbb{Z}$. $F = \{e_m\}_{m\in\mathbb{Z}}$, $e_m \in \operatorname{Sp} E$, $\forall m \in \mathbb{Z}$

$\Rightarrow \quad Sp F \subseteq Sp E$, $\{e_n\}$ is an o.n.b. $\Rightarrow$ it is complete $\Rightarrow \overline{Sp F} = L^2([-\pi, \pi])$

$L^2_{\mathbb{C}}([-\pi,\pi]) = \overline{SpF} \subseteq \overline{Sp E} \subseteq L^2([-\pi, \pi])$

$\Rightarrow \quad \overline{Sp E} = L^2([-\pi, \pi])$ that is, E is a complete seq. $\Leftrightarrow$ E o.n.b.

$\forall f \in L^2([-\pi, \pi])$

$$f = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Fourier series expansion

$$a_0 := \frac{1}{\sqrt{2\pi}}\left(f, \frac{1}{\sqrt{2\pi}}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

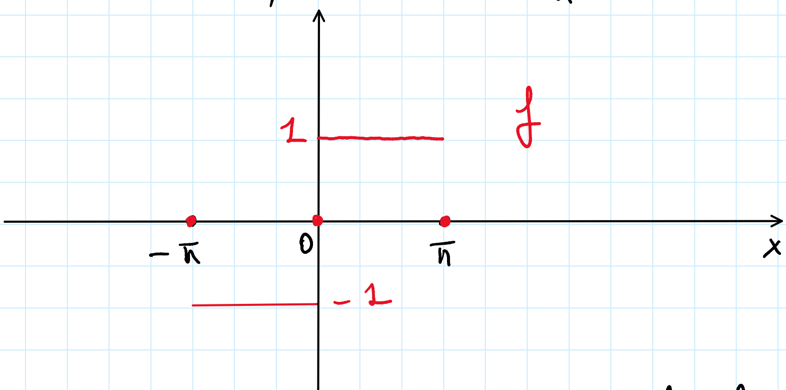
$$a_n := \frac{1}{\sqrt{\pi}}\left(f, \frac{1}{\sqrt{\pi}} \cos nx\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n := \frac{1}{\sqrt{\pi}}\left(f, \frac{1}{\sqrt{\pi}} \sin nx\right) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

"Fourier coefficients of f"

Exercise. Consider the square wave:

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 0, & x = 0 \text{ or } x = \pm\pi \\ 1, & 0 < x < \pi \end{cases}$$



f is an odd function

Compute the Fourier series of f.

Solution. Since f is an odd function

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{f(x)}_{\text{odd}}\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\cos nx}_{\text{even}}}_{\text{odd}}\,dx = 0, \quad \forall n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{odd}}\,\underbrace{\sin nx}_{\text{odd}}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\,dx = \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$b_n = \begin{cases} 0 & n = 2k \\ \dfrac{4}{n\pi} & n = 2k+1 \end{cases}$$

$$f = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin[(2k+1)x] = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin[(2k+1)x]}{2k+1}$$

Ex. 6, HMW2 Compute the Fourier series exp. of $f(x) = |x|$ on $[-\pi,\pi]$, (f is even)

LINEAR OPERATORS

From now on we study the properties of linear operators $T: X \to Y$, with X, Y normed spaces over the same scalar field $\mathbb{F}$.

T linear: $\forall \alpha,\beta \in \mathbb{F}$, $\forall x_1, x_2 \in X$, $T(\alpha x_1 + \beta x_2) =$
$= \alpha T(x_1) + \beta T(x_2)$

$$L(X,Y) = \{T: X \to Y,\ T \text{ linear}\}$$

Notation: $Tx := T(x)$

Theorem. X, Y normed spaces, $T : X \to Y$ linear.

Then the following conditions are equivalent:

1) T is bounded: $\exists\, C > 0 : \|Tx\|_Y \le C\|x\|_X,\ \forall x \in X$

2) $\exists\, C > 0 \ / \ \|Tx\|_Y \le C,\ \forall x \in X : \|x\|_X \le 1$

3) T is uniformly continuous on X

4) T is continuous on X

5) T is continuous at 0.

Proof. Trivially, $3) \Rightarrow 4) \Rightarrow 5)$

Let us show $5) \Rightarrow 2)$

Take $\varepsilon = 1$, since T is continuous at 0

$$\exists\, \delta > 0 : \forall x \in X,\ \|x\|_X < \delta \Rightarrow \|Tx\|_Y < 1$$

Now, $\forall w \in X$, $\|w\|_X \le 1$, $\Rightarrow \|\frac{\delta}{2} w\|_X = \frac{\delta}{2}\|w\|_X \le \frac{\delta}{2} < \delta$

T is linear

then $\|T(\frac{\delta}{2} w)\|_Y < 1$

$$\Updownarrow$$

$$\|\tfrac{\delta}{2} Tw\|_Y \iff \tfrac{\delta}{2}\|Tw\|_Y$$

So $\frac{\delta}{2}\|Tw\|_Y < 1 \iff \|Tw\|_Y < \frac{2}{\delta}$

Hence $\forall w : \|w\|_X \le 1 \Rightarrow \|Tw\|_Y < \frac{2}{\delta} = C$

$2) \Rightarrow 1)$ if $x = 0$ the boundedness estimate $\|Tx\|_Y \le C\|x\|_X$ becomes $0 = 0$, so satisfied $\forall C > 0$.

So consider $x \neq 0$, $y = \frac{x}{\|x\|_X}$ so that $\|y\| = 1$

by condition 2) $\exists\, C > 0 \mid \|Ty\|_Y \leq C$

that is $\left\| T\left(\frac{x}{\|x\|_X}\right) \right\|_Y \leq C$

$$\Updownarrow$$

$$\left\| \frac{1}{\|x\|_X} Tx \right\|_Y \leq C \iff \frac{1}{\|x\|_X} \|Tx\|_Y \leq C$$

$$\iff \|Tx\|_Y \leq C \|x\|_X, \qquad \forall x \in X.$$

Let us show 1) $\Rightarrow$ 3)

$\forall x, y \in X$, we have by assumption:

$$\|T(x-y)\|_Y \leq C \|x-y\|_X$$

$$\|T(x-y)\|_Y \overset{T \text{ lin.}}{=} \|Tx - Ty\|_Y$$

$\forall \varepsilon > 0$ choose $\delta = \frac{\varepsilon}{C}$, then $\forall x, y : \|x-y\|_X < \delta$

$$\|Tx - Ty\|_Y = \|T(x-y)\|_Y \leq C \|x-y\|_X < C\delta = \varepsilon$$

$\blacksquare$