LINEAR AND BOUNDED OPERATORS

$T: X \to Y$ linear, then t.f.c.a.e.:

1) $\exists\, C > 0$: $\|Tx\|_Y \le C \|x\|_X$, $\forall x \in X$
2) $\exists\, C > 0$: $\|Tx\|_Y \le C$, $\forall x$: $\|x\| \le 1$
3) T is unif. cont. on X
4) T is cont. on X
5) T is cont. at 0.

Remark. For $T \neq 0$ (0-mapping), if T is bounded then $\operatorname{Im} T$ is <u>never</u> bounded. In fact, if $T \neq 0$ then $\exists\, x \in X$: $Tx = q \neq 0$, so $\|Tx\|_Y = \|q\| \neq 0$ $\forall \alpha \in \mathbb{F}$, $\|T(\alpha x)\|_Y = \|\alpha Tx\|_Y = |\alpha|\,\|q\|$

So, if $|\alpha| \to +\infty$ $\Rightarrow$ $\|T(\alpha x)\|_Y \to +\infty$

$\{\alpha q\} \subset \operatorname{Im} T$ $\quad$ $\{\alpha q\}$ unbounded $\Rightarrow \operatorname{Im} T$ is unbounded.

Examples. 1) $X = \mathcal{C}_{\mathbb{F}}([0,1])$ with $\|\cdot\|_\infty$, $Y = \mathbb{F}$

Consider the operator $T: X \to Y$, $Tf = f(0)$

T is well defined

T is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall f, g \in \mathcal{C}_{\mathbb{F}}([0,1])$,

$$T(\alpha f + \beta g) \overset{\text{def.}}{=} (\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha Tf + \beta Tg$$

T is bounded: $\forall f \in \mathcal{C}_{\mathbb{F}}([0,1])$, $|Tf| = |f(0)| \le \|f\|_\infty$

2) Fix a sequence $c = (c_n) \in \ell^\infty$. For $1 \le p < \infty$, consider $T_c: \ell^p \to \ell^p$ defined by

(TC) $\qquad T_c((x_n)) = (c_n x_n)$, $\quad \forall\, x = (x_n) \in \ell^p$.

T_c is well defined: $\forall\, x = (x_n) \in \ell^p$

$$\|T_c x\|_{\ell^p}^p = \sum_{n=1}^{\infty} |c_n x_n|^p = \sum_{n=1}^{\infty} |c_n|^p |x_n|^p \leq \|c\|_{\ell^\infty}^p \|x\|_{\ell^p}^p < \infty$$

$$|c_n| \leq \sup_n |c_n| = \|c\|_{\ell^\infty}$$

So $T_c : \ell^p \to \ell^p$ is well defined and T_c is bounded $\|T_c x\|_{\ell^p} \leq \|c\|_{\ell^\infty} \|x\|_{\ell^p}$, $\forall x \in \ell^p$

T_c is linear: $\forall \alpha, \beta \in \mathbb{F}$, $\forall x = (x_n), y = (y_n) \in \ell^p$

$$T_c(\alpha x + \beta y) = \left(c_n(\alpha x_n + \beta y_n)\right) = (\alpha c_n x_n + \beta c_n y_n)$$
$$= \alpha(c_n x_n) + \beta(c_n y_n) = \alpha T_c x + \beta T_c y.$$

Definition. We denote by $B(X,Y)$ the space of all linear and bounded operators $T : X \to Y$.

Hence $B(X,Y) \subseteq L(X,Y) = \{T : X \to Y, \ T \text{ linear}\}$.

Lemma. $B(X,Y)$ is a normed space with

$$\|T\|_{B(X,Y)} = \sup\{\|Tx\|_Y : \|x\|_X \leq 1\} \qquad (A)$$

Proof. We know $B(X,Y) \subseteq L(X,Y)$ so in order to prove that it is a vector space, it is enough to show that $B(X,Y)$ is a subspace of $L(X,Y)$

$\forall \alpha, \beta \in \mathbb{F}$, $\forall T_1, T_2 \in B(X,Y)$, $\alpha T_1 + \beta T_2 \in L(X,Y)$

(recall that $L(X,Y)$ is a vector space)

$\alpha T_1 + \beta T_2$ is bounded:

$$\|(\alpha T_1 + \beta T_2)(x)\|_Y = \|\alpha T_1 x + \beta T_2 x\|_Y \leq |\alpha| \|T_1 x\|_Y + |\beta| \|T_2 x\|_Y$$

$T_1, T_2 \in B(X,Y) \Rightarrow \exists C_i > 0 : \|T_i x\|_Y \leq C_i \|x\|_X, \ \forall x \in X,$ $i = 1, 2$.

hence
$$|\alpha| \|T_1 x\|_Y + |\beta| \|T_2 x\|_Y \leq |\alpha| C_1 \|x\|_X + |\beta| C_2 \|x\|_X$$
$$= (|\alpha| C_1 + |\beta| C_2) \|x\|_X$$

$$\Rightarrow \|(\alpha T_1 + \beta T_2)(x)\|_Y \leq C \|x\|_X, \quad \forall x \in X$$

with $C = |\alpha| C_1 + |\beta| C_2$.

Let us show that $\|\cdot\|_{B(X,Y)} : B(X,Y) \to [0,+\infty)$ is a norm. First, it is well defined since by the equivalent definition of boundedness in 2) we have $\exists\, C>0 : \|Tx\|_Y \le C, \ \forall x: \|x\|_X \le 1$ hence $\{\|Tx\|_Y : \|x\| \le 1\}$ is bounded from above by C $\Rightarrow$ $\sup\{\|Tx\|_Y : \|x\| \le 1\} < \infty$.

For short, we write $\|T\| := \|T\|_{B(X,Y)}$ in (A)

1) $\|T\| \ge 0$ ok. since $\|Tx\|_Y \ge 0 \ \forall x$

2) $\|T\| = 0 \iff T$ is 0-mapping

assume first $\|T\| = 0$ hence $\|Tx\|_Y = 0, \forall x: \|x\|_X \le 1$ $\iff$ $Tx = 0, \ \forall x : \|x\|_X \le 1$

$\forall x \in X, x \ne 0,$ $y := \frac{x}{\|x\|}$ then $\|y\|_X = 1$ hence $Ty = 0$ $\Rightarrow$ $T\left(\frac{x}{\|x\|}\right) = 0 \iff \frac{1}{\|x\|} Tx = 0$

$\iff Tx = 0$

T is 0-mapping $\|Tx\|_Y = \|0\|_Y = 0, \forall x$

$\Rightarrow \|T\| = 0$.

3) $\forall \alpha \in \mathbb{F}$ $\|\alpha T\| = \sup\{\|\alpha Tx\|_Y : \|x\|_X \le 1\}$

if $\alpha \ne 0$ $\|\alpha Tx\|_Y = |\alpha| \, \|Tx\|_Y$

$\sup\{\underbrace{|\alpha|}_{>0} \|Tx\|_Y : \|x\|_X \le 1\} = |\alpha| \sup\{\|Tx\|_Y : \|x\|_X \le 1\}$

$= |\alpha| \, \|T\|$

if $\alpha = 0$ $\|\alpha T\| = \|0\| = 0 = \underbrace{|\alpha|}_{=0} \|T\|$

4) $\forall S, T \in B(X,Y),$ $\forall x \in X, \ \|x\|_X \le 1$

$\|(S+T)(x)\|_Y = \|Sx + Tx\|_Y \le \|Sx\|_Y + \|Tx\|_Y$

$\le \|S\| + \|T\|, \quad \forall x: \|x\|_X \le 1$

hence $\sup\{\|(S+T)(x)\|_Y : \|x\|_X \le 1\} \le \|S\| + \|T\|$

$\|S+T\| \le \|S\| + \|T\|$.

So $\left(B(X,Y), \|\cdot\|_{B(X,Y)}\right)$ is a normed space.

Equivalent definitions for $\|\cdot\|_{B(X,Y)}$

1) $\|T\|_{B(X,Y)} = \sup\{\|Tx\|_Y : \|x\|_X \le 1\}$

2) $\|T\|_{B(X,Y)} = \sup\{\|Tx\|_Y : \|x\|_X = 1\}$

3) $\|T\|_{B(X,Y)} = \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X}, \forall x \in X\setminus\{0\}\right\}$

4) $\|T\|_{B(X,Y)} = \inf\{C>0 : \|Tx\|_Y \le C, \forall x : \|x\|_X \le 1\}$

5) $\|T\|_{B(X,Y)} = \inf\{C>0 : \|Tx\|_Y \le C, \forall x : \|x\|_X = 1\}$

6) $\|T\|_{B(X,Y)} = \inf\{C>0 : \|Tx\|_Y \le C\|x\|_X, \forall x \in X\}$

Notation: if $Y = X$ we write $B(X) := B(X,X)$

Examples. 1) $T \in B(\mathcal{C}_{\mathbb{F}}([0,1]), \mathbb{F})$, $Tf = f(0)$

Compute $\|T\|$.

Solution. We already know: $|Tf| \le \|f\|_\infty$

so by 6) we obtain $\|T\| \le 1$

Consider the function $f(x) = 3$, $\forall x \in [0,1]$

$\Rightarrow$ $f \in \mathcal{C}_{\mathbb{F}}([0,1])$ and $|Tf| = |f(0)| = 3$

$\|f\|_\infty = 3$ $\Rightarrow$ $|Tf| = \|f\|_\infty$

$\Rightarrow$ $\|T\| = 1$.

2) X inner product space, fix $y \in X$. Define:

$Tx = (x, y)$, $\forall x \in X$.

$T : X \to \mathbb{F}$

- T is well defined

- T is linear $\quad T(\alpha x_1+\beta x_2) = (\alpha x_1+\beta x_2, y) = \alpha(x_1,y)+\beta(x_2,y)$
- T is bounded: $\quad |Tx| = |(x,y)| \overset{\text{Cauchy-Schwarz ineq.}}{\leq} \|x\|_X \|y\|_X$

So $\quad |Tx| \leq \underbrace{\|y\|_X}_{=C} \|x\|_X, \quad \forall x \in X$

- by the previous inequality, $\|T\| \leq \|y\|_X$

we want to show $\|T\| = \|y\|_X$

if $y=0$ $\quad Tx = (x,0) = 0, \quad \forall x \in X$ hence T is 0-map

and $\|T\| = 0 = \|y\|_X$

if $y \neq 0$, consider $x = \frac{y}{\|y\|}$ then $\|x\| = 1$

$$Tx = (x,y) = \left(\frac{y}{\|y\|}, y\right) = \frac{1}{\|y\|}(y,y) = \|y\|_X$$

So $\quad |Tx| = Tx = \|y\|_X$

So $\quad |Tx| = \|y\|_X \underbrace{\|x\|_X}_{=1} \quad \Rightarrow \quad \|T\| = \|y\|_X$.

Example. Fix $k \in \mathcal{C}_{\mathbb{C}}([a,b]\times[a,b])$ and define

$$M = \sup_{(x,y)\in[a,b]\times[a,b]} |k(x,y)| \overset{\text{Weierstrass Thrm.}}{=} \max_{(x,y)\in[a,b]\times[a,b]} |k(x,y)| = \|k\|_\infty$$

Consider the integral operator with kernel k:

$$Tf(x) = \int_a^b k(x,y)\, f(y)\, dy \qquad \text{(INT)}$$

k is called the kernel of T. Then,

$T: \mathcal{C}_{\mathbb{C}}([a,b]) \to \mathcal{C}_{\mathbb{C}}([a,b])$, T is linear and bounded, with $\|Tf\|_\infty \leq M(b-a)\|f\|_\infty, \quad \forall f \in \mathcal{C}_{\mathbb{C}}([a,b])$.

Solution. • T is well defined

$\forall f \in \mathcal{C}_{\mathbb{C}}([a,b])$, $\forall \varepsilon > 0$ $\quad k$ is continuous on $[a,b]\times[a,b] \Rightarrow k$ is uniformly continuous

so given $\varepsilon > 0$ above $\exists\, \delta > 0: \forall (x_1,y_1),(x_2,y_2) \in [a,b]\times[a,b]$ such that $d((x_1,y_1),(x_2,y_2)) < \delta \Rightarrow |k(x_1,y_1) - k(x_2,y_2)| < \varepsilon$

if $\quad y_1 = y_2 = y \qquad d((x_1, y), (x_2, y)) = \sqrt{|x_1 - x_2|^2} = |x_1 - x_2|$

So if $|x_1 - x_2| < \delta \implies |K(x_1, y) - K(x_2, y)| < \varepsilon$

$$
\begin{aligned}
|Tf(x_1) - Tf(x_2)| &= \left| \int_a^b K(x_1, y) f(y) dy - \int_a^b K(x_2, y) f(y) dy \right| \\
&\leq \int_a^b |K(x_1, y) - K(x_2, y)| \cdot |f(y)| \, dy \\
&< \varepsilon \int_a^b |f(y)| \, dy \\
&\leq \varepsilon \|f\|_\infty \int_a^b dy = \varepsilon (b-a) \|f\|_\infty
\end{aligned}
$$

if $f \equiv 0 \implies Tf \equiv 0$ so Tf is continuous

otherwise, if $f \neq 0$ $(\|f\|_\infty > 0)$ and

$|Tf(x_1) - Tf(x_2)| < C\varepsilon$ for $C = \|f\|_\infty (b-a) > 0$

- T is linear : $\forall \alpha, \beta \in \mathbb{C}, \forall f_1, f_2 \in \mathcal{C}([a,b])$

$$
\begin{aligned}
T(\alpha f_1 + \beta f_2) &= \int_a^b K(x,y) [\alpha f_1(y) + \beta f_2(y)] \, dy \\
&= \alpha \int_a^b K(x,y) f_1(y) dy + \beta \int_a^b K(x,y) f_2(y) dy \\
&= \alpha T f_1 + \beta T f_2
\end{aligned}
$$

- T is bounded: $\forall f \in \mathcal{C}([a,b])$

$$
\begin{aligned}
|Tf| = \left| \int_a^b K(x,y) f(y) \, dy \right| &\leq \int_a^b |K(x,y)| \cdot |f(y)| \, dy \\
&\leq \|K\|_\infty \|f\|_\infty \int_a^b dy = \underbrace{\|K\|_\infty (b-a)}_{C} \|f\|_\infty .
\end{aligned}
$$

Example. $K \in L^2([a,b] \times [a,b])$ and

$Tf(x) = \int_a^b K(x,y) f(y) \, dy \qquad$ for $f \in L^2([a,b])$.

Then $T : L^2([a,b]) \to L^2([a,b])$ well defined, linear and bounded, with $\|Tf\|_{L^2([a,b])} \leq \|K\|_{L^2([a,b] \times [a,b])} \|f\|_{L^2([a,b])}$.

Solution. $\forall f \in L^2([a,b])$

$$
|Tf(x)|^2 = \left| \int_a^b K(x,y) f(y) \, dy \right|^2 \leq \left(\int_a^b |K(x,y)| \cdot |f(y)| \, dy \right)^2
$$

Hölder's inea. with $p = q = 2$

$$
\leq \left(\left(\int_a^b |K(x,y)|^2 dy \right)^{1/2} \left(\int_a^b |f(y)|^2 dy \right)^{1/2} \right)^2
$$

$$= \int_a^b |K(x,y)|^2\,dy \quad \underbrace{\int_a^b |f(y)|^2\,dy}_{=\|f\|_2^2}$$

$K \in L^2([a,b]\times[a,b]) \Rightarrow |K(x,y)|^2 \in L^1([a,b]\times[a,b])$

by Fubini Theorem, $\int_a^b |K(x,y)|^2\,dy < \infty$

for a.e. $x \in [a,b]$, so

$$\int_a^b |Tf(x)|^2\,dx \leq \|f\|^2_{L^2([a,b])} \underbrace{\int_a^b \int_a^b |K(x,y)|^2\,dy\,dx}_{= \|K\|^2_{L^2([a,b]\times[a,b])}}$$

$\Rightarrow$ $Tf \in L^2([a,b])$ and

$$\|Tf\|_{L^2([a,b])} \leq \|K\|_{L^2([a,b]\times[a,b])} \|f\|_{L^2([a,b])}$$

- the linearity of T follows from the linearity of the integral. $\boxtimes$